1 Forecast Evaluation and Regression

Forecast errors and loss functions

$$e = y - \hat{y} \quad \text{(positive: underprediction; negative: overprediction).}$$

$$\text{MSPE} = \frac{1}{n}\sum_{i=1}^{n} e_i^2, \qquad \text{RMSPE} = \sqrt{\text{MSPE}}, \qquad \text{MAPE} = \frac{1}{n}\sum_{i=1}^{n} |e_i|.$$

- Smaller loss is better. MSPE penalizes large errors more heavily than MAPE.
  - Doubling an error quadruples its squared loss but only doubles its absolute loss.
- RMSPE and MAPE have the same units as y; MSPE has squared units.
  - If y is measured in dollars, RMSPE and MAPE are in dollars; MSPE is in dollars squared.
- One forecast may be unusually accurate or inaccurate. Averaging losses over multiple forecasts gives a better indication of typical performance. Compare models on the same targets, horizons, and available information for fairness.

Under squared-error loss, the optimal forecast is the conditional mean; under absolute-error loss, it is a conditional median.

- **Squared-error loss:** Let a be the forecast and $\mathcal{I}$ the available information.

  $$E[(y-a)^2 \mid \mathcal{I}] = E[y^2 \mid \mathcal{I}] - 2aE[y \mid \mathcal{I}] + a^2.$$

  Differentiate with respect to a and set equal to zero:

  $$-2E[y \mid \mathcal{I}] + 2a = 0 \quad \Rightarrow \quad a = E[y \mid \mathcal{I}].$$

  The second derivative is $2 > 0$, so this minimizes expected squared error.

- **Absolute-error loss:** For a continuous conditional distribution,

  $$\frac{d}{da}E[|y-a| \mid \mathcal{I}] = \text{P}(y < a \mid \mathcal{I}) - \text{P}(y > a \mid \mathcal{I}).$$

  Setting this equal to zero gives

  $$\text{P}(y < a \mid \mathcal{I}) = \text{P}(y > a \mid \mathcal{I}) = \frac{1}{2}.$$

  Thus, a conditional median minimizes expected absolute error: half the probability lies on each side.

Regression and prediction

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i, \qquad E(\varepsilon_i \mid x_i) = 0.$$

Then

$$E(y_i \mid x_i) = \beta_0 + \beta_1 x_i, \qquad \hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i.$$

OLS minimizes the sum of squared residuals.

Coefficient tests and confidence intervals

$$t = \frac{\hat{\beta}_j - b}{SE(\hat{\beta}_j)} \quad \text{for } H_0 : \beta_j = b.$$

Using the large-sample normal approximation, reject a two-sided null at the 5% level if $|t| > 1.96$. A 95% CI is

$$\hat{\beta}_j \pm 1.96\, SE(\hat{\beta}_j).$$

Exercise 1 Forecast evaluation. Two models produce the following one-quarter-ahead forecast errors for the same four quarters, in percentage points. Both use the same available information.

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Model A	1	−1	1	−1
Model B	0	0	0	3

1. Calculate MSPE and MAPE for each model. Which model does each criterion prefer?
2. Explain why the rankings differ. Would comparing only the first forecast be enough to judge the models?

Exercise 2 Regression. Hourly wages are modeled as

$$wage_i = \beta_0 + \beta_1 College_i + \varepsilon_i,$$

where $College_i = 1$ for college graduates and 0 for high-school graduates. A large-sample regression gives:

	Estimate	Standard error
Intercept	20	1
College	12	3

1. Interpret the coefficient on *College*. Calculate its t-statistic for $H_0 : \beta_1 = 0$. Is it significant at the 5% level in a two-sided test?
2. Predict the wage of a college graduate. If a new college graduate earns \$29 per hour, calculate and interpret the prediction error.

2 Forecast Intervals

A confidence interval describes uncertainty about an unknown parameter; a forecast interval describes uncertainty about a future outcome.

If the forecast error satisfies

$$e = y - \hat{y} \sim N(0, v^2),$$

then

$$\begin{aligned} 90\% \text{ forecast interval:} \quad & \hat{y} \pm 1.65v, \\ 95\% \text{ forecast interval:} \quad & \hat{y} \pm 1.96v. \end{aligned}$$

- v is the standard deviation of forecast errors. In practice, it is replaced by an estimate.
  - Given n past forecast errors at the same horizon, a sample-based estimate is

$$\hat{v} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(e_i - \bar{e})^2}, \qquad \bar{e} = \frac{1}{n}\sum_{i=1}^{n} e_i.$$

    This assumes past errors are representative of future errors.

- A coefficient standard error measures how precisely a regression coefficient is estimated.
  - In Exercise 2, $SE(\hat{\beta}_1) = 3$ measures uncertainty about the estimated wage difference of \$12, not the prediction error for a new person's wage.
- Higher coverage requires a wider interval.

Exercise 3 Forecast intervals. Using the regression from Exercise 2, the predicted hourly wage of a new college graduate is \$32. Assume the prediction error is normally distributed with mean zero and standard deviation $v = \$5$.

1. Calculate a 95% forecast interval for this person's hourly wage.
2. A student instead calculates $32 \pm 1.96(3)$, using the standard error of the *College* coefficient. Explain why this is incorrect.

3 Growth Rates and Real Values

Growth rates

Use log differences multiplied by 100 to calculate growth rates in percentage terms:

$$g_t = 100(\ln Y_t - \ln Y_{t-1}).$$

This approximates the ordinary percentage change when growth is small. The values of Y must be positive.

For quarterly data:

$$\begin{aligned} \text{QoQ growth (\%)} &= 100(\ln Y_t - \ln Y_{t-1}), \\ \text{Annualized QoQ growth (\%)} &= 400(\ln Y_t - \ln Y_{t-1}), \\ \text{YoY growth (\%)} &= 100(\ln Y_t - \ln Y_{t-4}). \end{aligned}$$

For monthly data:

$$\begin{aligned} \text{MoM growth (\%)} &= 100(\ln Y_t - \ln Y_{t-1}), \\ \text{Annualized MoM growth (\%)} &= 1200(\ln Y_t - \ln Y_{t-1}), \\ \text{YoY growth (\%)} &= 100(\ln Y_t - \ln Y_{t-12}). \end{aligned}$$

Annualization scales the latest period's log growth to one year. YoY growth compares the current level with its level one year earlier.

Nominal and real values

Nominal values are measured at current prices. Real values remove the effects of price changes.

$$\text{Real value in index-base-year dollars} = \frac{\text{Nominal value} \times 100}{\text{Price index}}.$$

To express a value in another year's dollars, replace 100 with that year's price index. Changing the common base year does not change the growth rate of real values.

Exercise 4 Growth rates. A quarterly price index takes the following values:

Quarter	Price index
2025 Q2	100.00
2026 Q1	102.00
2026 Q2	104.04

1. Using log differences, calculate QoQ inflation and annualized QoQ inflation for 2026 Q2.
2. Using log differences, calculate YoY inflation for 2026 Q2. Explain how it differs from annualized QoQ inflation.

Exercise 5 Nominal and real wages. Suppose hourly wages and the price index are:

Year	Nominal hourly wage	Price index (2020 = 100)
2020	\$20	100
2021	\$21	110

1. Calculate the real hourly wage in each year, expressed in 2020 dollars.

2. Calculate nominal and real wage growth from 2020 to 2021. Did purchasing power increase? Use unrounded values in your calculations.

Answers

Exercise 1 Forecast evaluation. Two models produce the following one-quarter-ahead forecast errors for the same four quarters, in percentage points. Both use the same available information.

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Model A	1	−1	1	−1
Model B	0	0	0	3

1. Calculate MSPE and MAPE for each model. Which model does each criterion prefer?

$$\text{MSPE}_A = \frac{1^2 + (-1)^2 + 1^2 + (-1)^2}{4} = 1,$$
$$\text{MAPE}_A = \frac{1 + 1 + 1 + 1}{4} = 1,$$
$$\text{MSPE}_B = \frac{0 + 0 + 0 + 3^2}{4} = 2.25,$$
$$\text{MAPE}_B = \frac{0 + 0 + 0 + 3}{4} = 0.75.$$

   MSPE prefers Model A; MAPE prefers Model B. MSPE is measured in squared percentage points, while MAPE is measured in percentage points.

2. Explain why the rankings differ. Would comparing only the first forecast be enough to judge the models?

   Model B has three perfect forecasts but one large error. Squaring that error gives a loss of $3^2 = 9$, compared with an absolute loss of 3. Thus, MSPE penalizes Model B's large error enough to reverse the ranking.

   Comparing only the first forecast is insufficient: Model B performs better in that quarter, but one outcome provides little evidence about typical performance.

Exercise 2 Regression. Hourly wages are modeled as

$$wage_i = \beta_0 + \beta_1 College_i + \varepsilon_i,$$

where $College_i = 1$ for college graduates and 0 for high-school graduates. A large-sample regression gives:

	Estimate	Standard error
Intercept	20	1
College	12	3

1. Interpret the coefficient on $College$. Calculate its t-statistic for $H_0 : \beta_1 = 0$. Is it significant at the 5% level in a two-sided test?

   College graduates have an estimated average hourly wage \$12 higher than high-school graduates. This difference is not necessarily causal.

$$t = \frac{\hat{\beta}_1 - 0}{SE(\hat{\beta}_1)} = \frac{12}{3} = 4.$$

   Since $|4| > 1.96$, reject H_0 at the 5% level using the large-sample normal approximation.

2. Predict the wage of a college graduate. If a new college graduate earns \$29 per hour, calculate and interpret the prediction error.

$$\widehat{wage} = 20 + 12(1) = 32.$$

The predicted hourly wage is \$32. The prediction error is

$$e = y - \hat{y} = 29 - 32 = -3.$$

The model overpredicts this person's hourly wage by \$3.

Exercise 3 Forecast intervals. Using the regression from Exercise 2, the predicted hourly wage of a new college graduate is \$32. Assume the prediction error is normally distributed with mean zero and standard deviation $v = \$5$.

1. Calculate a 95% forecast interval for this person's hourly wage.

$$\hat{y} \pm 1.96v = 32 \pm 1.96(5) = 32 \pm 9.80.$$

The 95% forecast interval is [\$22.20, \$41.80].

2. A student instead calculates $32 \pm 1.96(3)$, using the standard error of the *College* coefficient. Explain why this is incorrect.

The standard error of 3 measures uncertainty about the estimated coefficient on *College*, not the prediction error for a new person's wage. The forecast interval requires the forecast-error standard deviation, $v = 5$.

Exercise 4 Growth rates. A quarterly price index takes the following values:

Quarter	Price index
2025 Q2	100.00
2026 Q1	102.00
2026 Q2	104.04

1. Using log differences, calculate QoQ inflation and annualized QoQ inflation for 2026 Q2.

$$\text{QoQ inflation} = 100(\ln 104.04 - \ln 102) \approx 1.98\%.$$

$$\text{Annualized QoQ inflation} = 400(\ln 104.04 - \ln 102) \approx 7.92\%.$$

The annualized log growth rate is four times the quarterly log growth rate.

2. Using log differences, calculate YoY inflation for 2026 Q2. Explain how it differs from annualized QoQ inflation.

$$\text{YoY inflation} = 100(\ln 104.04 - \ln 100) \approx 3.96\%.$$

YoY inflation measures log growth over the past year. Annualized QoQ inflation scales the latest quarter's log growth by four. They cover different comparisons and need not be equal.

Exercise 5 Nominal and real wages. Suppose hourly wages and the price index are:

Year	Nominal hourly wage	Price index (2020 = 100)
2020	\$20	100
2021	\$21	110

1. Calculate the real hourly wage in each year, expressed in 2020 dollars.

$$\text{2020 real wage} = \frac{20 \times 100}{100} = \$20.00,$$
$$\text{2021 real wage} = \frac{21 \times 100}{110} \approx \$19.09.$$

Both wages are expressed in 2020 dollars.

2. Calculate nominal and real wage growth from 2020 to 2021 using both ordinary percentage changes and log differences multiplied by 100. Did purchasing power increase? Use unrounded values in your calculations.

Method 1: Ordinary percentage changes

$$\text{Nominal wage growth} = 100\left(\frac{21}{20} - 1\right) = 5.00\%.$$

$$\text{Real wage growth} = 100\left(\frac{21 \times 100/110}{20} - 1\right) \approx -4.55\%.$$

Method 2: Log differences

$$\text{Nominal wage log growth} = 100 \ln\left(\frac{21}{20}\right) \approx 4.88\%.$$

$$\text{Real wage log growth} = 100 \ln\left(\frac{21 \times 100/110}{20}\right) \approx -4.65\%.$$

Equivalently, real wage log growth equals nominal wage log growth minus price-index log growth:

$$100 \ln\left(\frac{21}{20}\right) - 100 \ln\left(\frac{110}{100}\right) \approx 4.88 - 9.53 = -4.65\%.$$

Both methods show that purchasing power fell: prices rose faster than nominal wages. Log growth approximates ordinary percentage growth, so the two methods give slightly different numerical values.