1 Deterministic Trends and Forecasting

Time series components

$$y_t = T_t + S_t + C_t.$$

- T_t (trend): long-term movement.
- S_t (seasonal): patterns that repeat at fixed times of the year.
- C_t (cycle): remaining fluctuations, which need not follow a fixed calendar pattern.

For example, consider monthly ice cream sales. A growing neighborhood raises sales over time (T_t); summer raises sales each year (S_t); an economic downturn temporarily lowers sales (C_t). In one July, sales might be

$$\underbrace{1{,}200}_{y_t} = \underbrace{1{,}000}_{T_t} + \underbrace{400}_{S_t} - \underbrace{200}_{\text{downturn}}.$$

All values are in scoops.

Seasonal adjustment subtracts the estimated seasonal component from the original series:

$$y_t^{SA} = y_t - \widehat{S}_t.$$

In this example, removing the usual July boost gives $1{,}200 - 400 = 800$ scoops. This removes seasonality, but retains the trend and cyclical component. For the rest of this section, let y_t denote the seasonally adjusted series, so $y_t = T_t + C_t$.

Linear and quadratic trends

Linear	**Quadratic**
$T_t = \beta_0 + \beta_1 t$	$T_t = \beta_0 + \beta_1 t + \beta_2 t^2$
A straight line: the change each period is constant. $T_{t+1} - T_t = \beta_1.$	A curve: the change each period varies with time. $T_{t+1} - T_t = \beta_1 + \beta_2(2t+1).$
$\beta_1 > 0$: upward trend.	$\beta_2 > 0$: curves upward.
$\beta_1 < 0$: downward trend.	$\beta_2 < 0$: curves downward.

Given estimated coefficients, calculate the fitted trend $\widehat{T}_t$. Detrending means subtracting it:

$$\widehat{C}_t = y_t - \widehat{T}_t.$$

A positive residual means the observation is above its fitted trend; a negative residual means it is below.

Changing the time index

Changing the starting value of the time index may change the coefficients, but not fitted values, residuals, or trend forecasts.

For a linear trend, let $s = t + k$ label the same observation using a different starting value. Since $t = s - k$,

$$T_t = \beta_0 + \beta_1 t = \beta_0 + \beta_1(s-k) = \underbrace{(\beta_0 - k\beta_1)}_{b_0} + \underbrace{\beta_1}_{b_1} s.$$

Thus, writing the trend as $b_0 + b_1 s$ requires

$$b_0 = \beta_0 - k\beta_1, \qquad b_1 = \beta_1.$$

The intercept generally changes; the slope stays the same. The coefficients need not be identical for the two representations to give the same trend value at each date.

For a quadratic trend, the same substitution gives

$$\begin{aligned} T_t &= \beta_0 + \beta_1(s-k) + \beta_2(s-k)^2 \\ &= \underbrace{(\beta_0 - k\beta_1 + k^2\beta_2)}_{b_0} + \underbrace{(\beta_1 - 2k\beta_2)}_{b_1} s + \underbrace{\beta_2}_{b_2} s^2. \end{aligned}$$

Here both b_0 and b_1 generally differ from β_0 and β_1, while $b_2 = \beta_2$.

Each coefficient choice under one index has an equivalent choice under the other, producing the same fitted values and sum of squared residuals. Therefore, OLS gives the same fitted trend and residuals.

Forecasting the trend and the cycle

Consider a linear trend model:

$$y_t = \beta_0 + \beta_1 t + C_t.$$

We absorb any constant mean of C_t into the intercept, so $E(C_t) = 0$. The forecast is

$$\widehat{y}_{t+h|t} = \widehat{\beta}_0 + \widehat{\beta}_1(t+h) + \widehat{C}_{t+h|t}.$$

A zero mean does not necessarily imply a zero forecast: if deviations persist, today's cycle helps predict tomorrow's cycle.

Ignoring coefficient estimation uncertainty,

$$e_{t+h|t} \approx C_{t+h} - \widehat{C}_{t+h|t}.$$

If C_t is i.i.d. $N(0, \sigma_C^2)$, then $\widehat{C}_{t+h|t} = 0$, and a 90% forecast interval is

$$\widehat{\beta}_0 + \widehat{\beta}_1(t+h) \ \pm \ 1.65\sigma_C.$$

Under this assumption, the trend determines the center, while unpredictable cyclical fluctuations determine the width.

Exercise 1 Trend, cycle, and forecasting.

A seasonally adjusted series follows

$$y_t = \underbrace{\beta_0 + \beta_1 t}_{T_t} + C_t.$$

The estimated coefficients are $\widehat{\beta}_0 = 20$ and $\widehat{\beta}_1 = 3$, so the fitted trend is

$$\widehat{T}_t = 20 + 3t.$$

At $t = 4$, the observed value is $y_4 = 36$. Assume C_t is i.i.d. $N(0, 2^2)$ and ignore coefficient estimation uncertainty.

1. Calculate the fitted trend and residual at $t = 4$. Interpret the residual.

2. Forecast y_5 and construct a 90% forecast interval. Why do we not add the current residual to the trend forecast?

3. Another researcher uses $s = t + 10$. Rewrite the fitted trend in terms of s and verify that the forecast for the same future period is unchanged.

2 Seasonal Dummies and Forecasting

Modeling seasonality

Consider a quarterly series with no trend:

$$y_t = S_t + C_t.$$

A deterministic seasonal pattern repeats each year. Define $D_{jt} = 1$ if observation t is in quarter j, and 0 otherwise. Then

$$y_t = \underbrace{\gamma_1 D_{1t} + \gamma_2 D_{2t} + \gamma_3 D_{3t} + \gamma_4 D_{4t}}_{S_t} + C_t.$$

With a cycle whose mean is zero in each quarter, γ_j is the mean of y_t in quarter j. For example, in Q3, $D_{3t} = 1$ and all other dummies are zero, so $S_t = \gamma_3$.

Here the quarterly means include the overall level of the series; they need not sum to zero.

Two equivalent representations

We can use either four quarterly dummies with no intercept, or an intercept and three quarterly dummies:

$$y_t = \gamma_1 D_{1t} + \gamma_2 D_{2t} + \gamma_3 D_{3t} + \gamma_4 D_{4t} + C_t,$$
$$y_t = b_0 + b_2 D_{2t} + b_3 D_{3t} + b_4 D_{4t} + C_t.$$

Since exactly one quarterly dummy equals one,

$$D_{1t} = 1 - D_{2t} - D_{3t} - D_{4t}.$$

Substituting into the first model gives

$$S_t = \gamma_1 + (\gamma_2 - \gamma_1) D_{2t} + (\gamma_3 - \gamma_1) D_{3t} + (\gamma_4 - \gamma_1) D_{4t}.$$

Therefore,

$$b_0 = \gamma_1, \qquad b_j = \gamma_j - \gamma_1, \quad j = 2, 3, 4.$$

- b_0 is the Q1 mean; each b_j is the difference between quarter j and Q1.
- Both representations give the same fitted values, residuals, and forecasts.
- Do not include an intercept and all four dummies: their sum already equals the intercept regressor, so the coefficients are not uniquely determined.

Forecasting seasonal components

The future quarter is known, so future dummy values are known. If $t + h$ is in quarter j,

$$\widehat{S}_{t+h} = \widehat{\gamma}_j, \qquad \widehat{y}_{t+h|t} = \widehat{\gamma}_j + \widehat{C}_{t+h|t}.$$

If C_t is i.i.d. with mean zero, the forecast is simply $\widehat{\gamma}_j$.

For monthly data, use 12 dummies without an intercept, or an intercept and 11 dummies.

Exercise 2 Seasonal dummies.

Quarterly sales, measured in thousands of dollars, follow a model with no trend:

$$y_t = \gamma_1 D_{1t} + \gamma_2 D_{2t} + \gamma_3 D_{3t} + \gamma_4 D_{4t} + C_t.$$

The estimated coefficients are

$$\widehat{\gamma}_1 = 80, \qquad \widehat{\gamma}_2 = 100, \qquad \widehat{\gamma}_3 = 140, \qquad \widehat{\gamma}_4 = 120.$$

1. Rewrite the fitted seasonal component using an intercept and dummies for Q2, Q3, and Q4. Interpret the intercept and the Q3 coefficient.

2. Actual sales in 2026 Q3 are $y_t = 150$. Calculate $\widehat{S}_t$ and $\widehat{C}_t$.

3. Assume C_t is i.i.d. with mean zero. Standing in 2026 Q3, forecast sales for 2026 Q4 and 2027 Q1.

4. A student includes an intercept and all four quarterly dummies in the regression. Explain the problem.

3 A First Look at Forecasting with Lags

Using the current value to forecast

For a series with no growth trend or seasonality,

$$y_t = \mu + C_t.$$

If deviations from the mean persist, today's deviation can help predict tomorrow's. Using the sample mean $\bar{y}$, a simple one-step forecast is

$$\underbrace{\widehat{y}_{T+1|T}}_{\text{forecast}} = \underbrace{\bar{y}}_{\text{mean}} + \underbrace{\widehat{b}(y_T - \bar{y})}_{\text{predicted cycle}} .$$

Here T is the last observed period and $\widehat{b}$ is an estimated regression slope. When $0 < \widehat{b} < 1$, part of the current deviation is predicted to persist, so the forecast lies between y_T and $\bar{y}$.

Exercise 3 A one-step forecast.

Inflation follows $y_t = \mu + C_t$. Suppose $\bar{y} = 3$, $y_T = 7$, and $\widehat{b} = 0.75$. Inflation is measured in percent.

1. Calculate the predicted cyclical component and the forecast of next period's inflation.

2. Is the forecast closer to the mean than the current value? Explain briefly.

Answers

Exercise 1 Trend, cycle, and forecasting.

A seasonally adjusted series follows

$$y_t = \underbrace{\beta_0 + \beta_1 t}_{T_t} + C_t.$$

The estimated coefficients are $\widehat{\beta}_0 = 20$ and $\widehat{\beta}_1 = 3$, so the fitted trend is

$$\widehat{T}_t = 20 + 3t.$$

At $t = 4$, the observed value is $y_4 = 36$. Assume C_t is i.i.d. $N(0, 2^2)$ and ignore coefficient estimation uncertainty.

1. Calculate the fitted trend and residual at $t = 4$. Interpret the residual.

   $$\widehat{T}_4 = 20 + 3(4) = 32, \qquad \widehat{C}_4 = 36 - 32 = 4.$$

   The observation is 4 units above its fitted trend.

2. Forecast y_5 and construct a 90% forecast interval. Why do we not add the current residual to the trend forecast?

   Since the cycle is i.i.d. with mean zero, $\widehat{C}_{5|4} = 0$. Thus,

   $$\widehat{y}_{5|4} = 20 + 3(5) = 35.$$

   The 90% forecast interval is

   $$35 \pm 1.65(2) = 35 \pm 3.3 = [31.7,\ 38.3].$$

   Under the i.i.d. assumption, the current positive residual does not predict a positive cycle next period. We therefore do not carry it forward into the forecast.

3. Another researcher uses $s = t + 10$. Rewrite the fitted trend in terms of s and verify that the forecast for the same future period is unchanged.

   Since $t = s - 10$,

   $$\widehat{T} = 20 + 3(s - 10) = -10 + 3s.$$

   The future period $t = 5$ corresponds to $s = 15$:

   $$\widehat{y}_{5|4} = -10 + 3(15) = 35.$$

   The intercept changes from 20 to -10, but the forecast is unchanged.

Exercise 2 Seasonal dummies.

Quarterly sales, measured in thousands of dollars, follow a model with no trend:

$$y_t = \gamma_1 D_{1t} + \gamma_2 D_{2t} + \gamma_3 D_{3t} + \gamma_4 D_{4t} + C_t.$$

The estimated coefficients are

$$\widehat{\gamma}_1 = 80, \qquad \widehat{\gamma}_2 = 100, \qquad \widehat{\gamma}_3 = 140, \qquad \widehat{\gamma}_4 = 120.$$

1. Rewrite the fitted seasonal component using an intercept and dummies for Q2, Q3, and Q4. Interpret the intercept and the Q3 coefficient.

   $$\widehat{S}_t = 80 + 20D_{2t} + 60D_{3t} + 40D_{4t}.$$

   The intercept is the estimated Q1 mean: \$80,000. The Q3 coefficient means that average Q3 sales are \$60,000 higher than Q1 sales.

2. Actual sales in 2026 Q3 are $y_t = 150$. Calculate $\widehat{S}_t$ and $\widehat{C}_t$.

   In Q3, $D_{3t} = 1$ and the other dummies are zero.

   $$\widehat{S}_t = 140, \qquad \widehat{C}_t = 150 - 140 = 10.$$

   Sales are \$10,000 above the estimated Q3 mean.

3. Assume C_t is i.i.d. with mean zero. Standing in 2026 Q3, forecast sales for 2026 Q4 and 2027 Q1.

   The future cycle is forecast as zero:

   $$\widehat{y}_{2026Q4|2026Q3} = 120, \qquad \widehat{y}_{2027Q1|2026Q3} = 80.$$

   The forecasts are \$120,000 and \$80,000. The current positive residual does not change these forecasts under the i.i.d. assumption.

4. A student includes an intercept and all four quarterly dummies in the regression. Explain the problem.

   For every observation,

   $$D_{1t} + D_{2t} + D_{3t} + D_{4t} = 1.$$

   Thus, the regressors are perfectly collinear. Adding a constant to the intercept and subtracting it from every dummy coefficient leaves fitted values unchanged, so the coefficients are not uniquely determined. Omit one dummy or the intercept.

Exercise 3 A one-step forecast.

Inflation follows $y_t = \mu + C_t$. Suppose $\bar{y} = 3$, $y_T = 7$, and $\widehat{b} = 0.75$. Inflation is measured in percent.

1. Calculate the predicted cyclical component and the forecast of next period's inflation.

$$\widehat{C}_{T+1|T} = 0.75(7 - 3) = 3, \qquad \widehat{y}_{T+1|T} = 3 + 3 = 6\%.$$

   The predicted cycle is 3 percentage points.

2. Is the forecast closer to the mean than the current value? Explain briefly.

   Yes. The current value is 4 percentage points above the mean, while the forecast is only 3 percentage points above it. The model predicts that 75% of the current deviation will persist.