1 Review: Key Equations

(By assume constant saving rate, we are able to skip the consumer problem)

- Production function: $Y_t = A_t F(K_t, N_t)$
- Resource constraint: $Y_t = C_t + I_t$
- Law of motion: $K_{t+1} = I_t + (1-\delta)K_t$
- Savings: $I_t = sY_t$
- FOCs: $w_t = AF_N(K_t, N_t)$, $r_t = AF_K(K_t, N_t)$

2 A Streamline Representations

2.1 Solow Model: Capital per Worker

By combining the motion of capital $K_t t + 1 = I_t + (1-\delta)K_t$ and investment in terms of savings $I_t = sY_t$, we are able to write the motion of capital in terms of capital per worker:

$$K_{t+1} = I_t + (1-\delta)K_t$$

By $I_t = sY_t$:

$$K_{t+1} = sY_t + (1-\delta)K_t$$

By the production function:

$$K_{t+1} = sAF(K_t, N_t) + (1-\delta)K_t$$

Divided by N_t for both sides:

$$\frac{K_{t+1}}{N_t} = \frac{sAF(K_t, N_t)}{N_t} + (1-\delta)\frac{K_t}{N_t}$$

Rearrange,

$$\frac{K_{t+1}}{N_{t+1}}\frac{N_{t+1}}{N_t} = \frac{sAF(K_t, N_t)}{N_t} + (1-\delta)\frac{K_t}{N_t}$$

Apply the **homogeneity of the production function**: $\frac{AF(K_t,N_t)}{N_t} = AF(\frac{K_t}{N_t}, \frac{N_t}{N_t}) = AF(k_t, 1)$, and define $f(k_t) \equiv F(k_t, 1)$. Note that we equate $N_{t+1} = N_t$ at this stage, then we are able to rewrite the equation above as following:

$$k_{t+1} = sAf(k_t) + (1-\delta)k_t$$

2.2 Solow Model: Others

Let's express other important variables in terms of capital per worker:

Output: By $y_t \equiv \frac{Y_t}{N_t}$,

$$y_t \equiv \frac{Y_t}{N_t} \equiv \frac{AF(K_t, N_t)}{N_t} = AF(k_t, 1) \equiv Af(k_t)$$

Consumption: By $c_t \equiv \frac{C_t}{N_t}$ and $C_t = (1-s)Y_t$:

$$c_t \equiv \frac{C_t}{N_t} = \frac{(1-s)Y_t}{N_t} = (1-s)y_t = (1-s)Af(k_t)$$

Investment: By $i_t \equiv \frac{I_t}{N_t}$ and $I_t = sY_t$,

$$i_t \equiv \frac{I_t}{N_t} = \frac{sY_t}{N_t} = sy_t = sAf(k_t)$$

Wage and Capital Return:

Math preparations:

- Euler Theorem: Functions that are homogeneous of degree ρ satisfies:

$$\rho F(K_t, N_t) = F_K(K_t, N_t)K_t + F_N(K_t, N_t)N_t$$
$$F_K(\gamma K_t, \gamma N_t) = \gamma^{\rho-1} F_K(K_t, N_t)$$

- FOCs:

$$w_t = AF_N(K_t, N_t)$$
$$r_t = AF_K(K_t, N_t)$$

Capital return:

- Step 1: Let $\rho = 1$ and apply Euler Theorem,

$$F_K(\gamma K_t, \gamma N_t) = F_K(K_t, N_t)$$

- Step 2: Let $\gamma = \frac{1}{N_t}$:

$$f'(k_t) = F_K(k_t, 1) = F_K(\frac{K_t}{N_t}, \frac{N_t}{N_t}) = F_K(K_t, N_t)$$

- Step 3: Plug in the FOC:

$$r_t = Af'(k_t)$$

Wage:

- Step 1: Let $\rho = 1$ and apply Euler Theorem:

$$F(K_t, N_t) = F_K(K_t, N_t)K_t + F_N(K_t, N_t)N_t$$

- Step 2: Divide both sides by N_t, and apply the homogeneity of production function:

$$\frac{F(K_t, N_t)}{N_t} = \frac{F_K(K_t, N_t)K_t}{N_t} + \frac{F_N(K_t, N_t)N_t}{N_t}$$
$$F(\frac{K_t}{N_t}, \frac{N_t}{N_t}) = F_K(K_t, N_t)\frac{K_t}{N_t} + F_N(K_t, N_t)\frac{N_t}{N_t}$$

- Step 3: Multiple both sides by A. By $f(k_t) = F(k_t, 1)$ and FOCs: $w_t = AF_N(K_t, N_t)$, $r_t = AF_K(K_t, N_t) = Af'(k_t)$:

$$AF(\frac{K_t}{N_t}, \frac{N_t}{N_t}) = AF_K(K_t, N_t)\frac{K_t}{N_t} + AF_N(K_t, N_t)\frac{N_t}{N_t}$$
$$Af(k_t) = Af'(k_t)k_t + w_t$$

- Step 4: Rearrange:

$$w_t = Af(k_t) - Af'(k_t)k_t$$

3 Steady State: A Graphic Analysis

3.1 Steady State

An economy will eventually converge to the steady state regardless of any initial starting point. At the steady state **the capital per worker** does not change over time.

3.2 Derive the Steady State

Since at the steady state, the capital per worker does not change over time, we are able to derive the steady state level of capital per worker by the law of motion of k_t(other variables could be derived by k^* given some initial values):

$$k_{t+1} = sAf(k_t) + (1-\delta)k_t$$

Since the capital per worker doesn't change over time, we are able to drop the time script:

$$k^* = sAf(k^*) + (1-\delta)k^*$$

Intuition: New investment equals depreciating capital, so the capital per worker at steady state doesn't change.

$$sAf(k^*) = \delta k^*$$

3.3 Important Properties

Graph k_{t+1} against k_t, the slope of the curve is just the first order derivative of the law of motion:

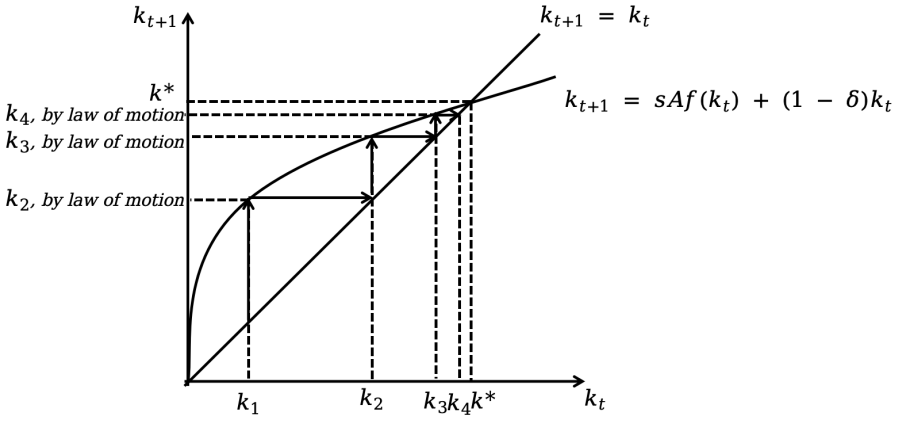
$$\frac{dk_{t+1}}{dk_t} = sAf'(k_t) + (1-\delta)$$

Important properties:

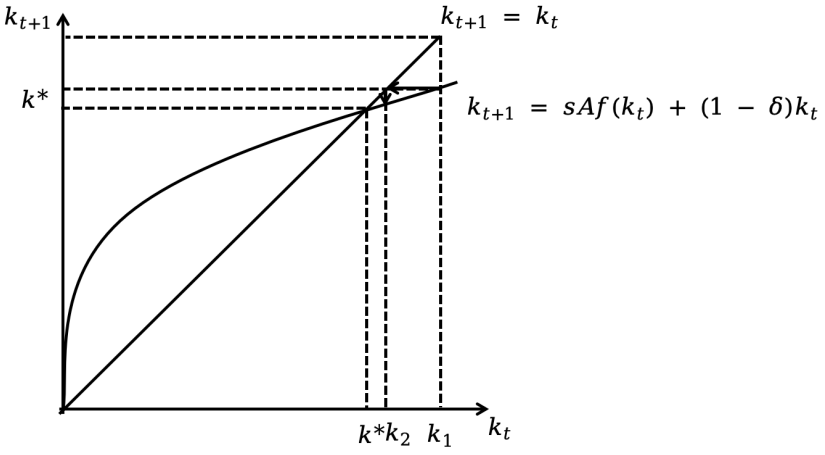
- k_{t+1} is monotonically increasing with k_t: $\frac{dk_{t+1}}{dk_t} = sAf'(k_t) + (1-\delta) > 0$. See below for the further explanation.
- $f(k)$ is monotonically increasing and is concave: $f' > 0,\ f'' < 0$
- Inada conditions: $\lim_{k_t \to 0} f'(k_t) = \infty$ and $\lim_{k_t \to \infty} f'(k_t) = 0$

3.4 Converge to the Steady State

Case 1: $k_t < k^*$:



Case 2: $k_t > k^*$



4 Cobb-Douglas Example

4.1 Steady State with Cobb-Douglas Production Function

At steady state:

General Case	Cobb-Douglas Case
$k^* = sAf(k^*) + (1-\delta)k^*$	$k^* = sAk^{*\alpha} + (1-\delta)k^*$
$y^* = Af(k^*)$	$y^* = Ak^{*\alpha}$
$c^* = (1-s)Af(k^*)$	$c^* = (1-s)Ak^{*\alpha}$
$i^* = sAf(k^*)$	$i^* = sA(k^*)^\alpha$
$r^* = Af'(k^*)$	$r^* = A\alpha(k^*)^{\alpha-1}$
$w^* = Af(k^*) - Af'(k^*)k^*$	$w^* = (1-\alpha)A(k^*)^\alpha$

Solve for k^*:

$$k^* = sAk^{*\alpha} + (1-\delta)k^*$$

$$k^* = \left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}$$

Then,

$$y^* = A^{\frac{1}{1-\alpha}} \left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

$$c^* = (1-s)A^{\frac{1}{1-\alpha}} \left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

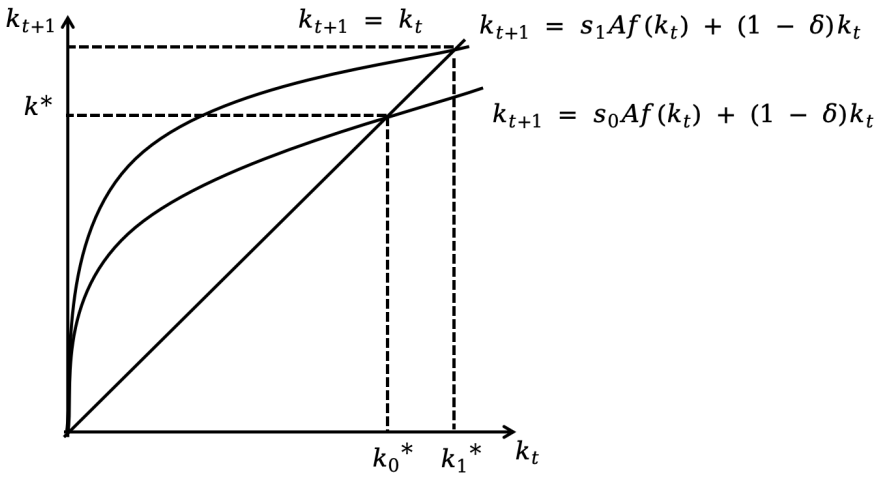
$$i^* = sA^{\frac{1}{1-\alpha}} \left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

$$r^* = \frac{\alpha\delta}{s}$$

$$w^* = (1-\alpha)A^{\frac{1}{1-\alpha}} \left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

4.2 An increase in the saving rate: $s_1 > s_0$

Law of Motion for Capital: By $k_{t+1} = sf(k_t) + (1-\delta)k_t$, $s \uparrow \rightarrow k_{t+1} \uparrow$ for each k_t:



Capital, Output, Wage, Capital Return:

- $k^* = \left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}$, so $k_1^* > k_0^*$
- $y^* = A^{\frac{1}{1-\alpha}} \left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}}$, so $y_1^* > y_0^*$
- $w^* = (1-\alpha)A^{\frac{1}{1-\alpha}} \left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}}$, so $w_1^* > w_0^*$
- $r^* = \frac{\alpha\delta}{s}$, so $r_1^* < r_0^*$

Take capital per worker as an example,

- Steady state: By $k^* = sAf(k^*) + (1-\delta)k^*$, since $s_2 > s_1$, we have $k_1^* > k_0^*$. The new steady-state capital is greater than the old one.
- Why no jump: Intuitively, capital per worker is accumulated gradually. Specifically, when assuming continuous time, time derivative of of capital is:

$$\frac{dk_t}{dt} = \lim_{\Delta t \to 0} \frac{k_{t+\Delta t} - k_t}{\Delta t}$$

Since we are working on discrete-time version, we can approximate it by:

$$\frac{dk_t}{dt} \approx k_{t+1} - k_t = \dot{k}$$

, where $\dot{k}$ is the change of capital per worker in a short time. Plug in the law of motion $k_{t+1} = sAf(k_t) + (1-\delta)k_t$:

$$\dot{k} = sAf(k_t) + (1-\delta)k_t - k_t$$

Then,

$$\dot{k} = sAf(k_t) - \delta k_t$$

Since $\dot{k}$ is a well-defined first-order differential equation, the derivative $\dot{k}$ exists for all t. So accumulation of k is a continuous process over time.

Since w, r and y are functions of k(no s in the functions), there are also no jumps for these three cases.

Consumption, Investment

- $c^* = (1-s)Af(k^*)$, the relationship between c_1^* and c_0^* is ambiguous. Could be $c_1^* > c_0^*$,$c_1^* < c_0^*$, or $c_1^* = c_0^*$. See the general example below:

$$c_1^* < c_0^* \Leftrightarrow (1-s_0)Af(k_0^*) = (1-s_1)Af(k_1^*)$$

$$\Leftrightarrow \frac{1-s_0}{1-s_1} = \frac{f(k_1^*)}{f(k_0^*)}$$

, which is compatible since $f(k_1^*) > f(k_0^*)$ and $s_0 < s_1$. But there will always be a drop as soon as the change occurs, since consumption is a function of s and s changes immediately.

Golden Rule: Call the level of saving that maximizes steady-state consumption the "golden rule" level of saving, s^{gr}

- Steady state (long-run) consumption is

$$c^* = (1-s)Af(k^*)$$

- The FOC with respect to saving rate

$$\frac{dc^*}{ds} = -Af(k^*) + (1-s)Af'(k^*)\frac{dk^*}{ds} = 0$$

- By $\frac{dk^*}{ds} = \frac{Af(k^*)}{\delta - sAf'(k^*)}$, the capital per worker satisfying the above FOC is

$$Af'(k^{gr}) = \delta$$

– One can derive the Golden rule level of saving (s^{gr}) from the following equation

$$s^{gr} = \frac{\delta k^{gr}}{Af(k^{gr})}$$

- $i^* = sA^{\frac{1}{1-\alpha}} \left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}}$, so $i_1^* > i_0^*$. There is a jump as soon as the change occurs since investment is a function of s and s changes immediately.

5 Exercise: An immediate one-time increase in A(Cobb-Douglas Case)

5.1 How will k, y, c, i, r, w change

5.2 Draw graphs showing how these variables change

6 Solution: An immediate one-time increase in A

Steps:

6.1 How will k, y, c, i, r, w change (Cobb-Douglas Case)

- Step 1: Analyze the steady state
  - $k^* = \left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}$: $A \uparrow \rightarrow k^* \uparrow$
  - $y^* = Af(k^*)$: $A \uparrow \& k^* \uparrow \rightarrow A \uparrow \& f(k^*) \uparrow \rightarrow y^* \uparrow$
  - $c^* = (1-s)Af(k^*)$: $A \uparrow \& k^* \uparrow \rightarrow A \uparrow \& f(k^*) \uparrow \rightarrow c^* \uparrow$
  - $i^* = sAf(k^*)$: $A \uparrow \& k^* \uparrow \rightarrow A \uparrow \& f(k^*) \uparrow \rightarrow i^* \uparrow$
  - $r^* = Af'(k^*)$: For Cobb-Douglas Case: $r^* = \frac{\alpha\delta}{s}$, $r_1^* = r_0^*$. Capital return remains unchanged.
  - $w^* = Af(k^*) - Af'(k^*)k^*$:For Cobb-Douglas case, $w^* = (1-\alpha)A^{\frac{1}{1-\alpha}}\left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}}$, so $w_1^* > w_0^*$
- Step 2: Find if there is a jump/drop
  - k changes gradually.
  - Other variables are all functions of A(note that not the steady state one), so there are jumps/drops for these variables

6.2 Draw graphs showing how these variables change

See the lecture slides.