1 Augmented Solow Growth Model

1.1 Aggregate Production Function

$$Y_t = AF(K_t, Z_t N_t)$$

The "efficiency" Z_t:

- Z_t is exogenous
- $Z_t N_t$: The effective number of workers, or the efficiency units of labor
- $F_Z > 0$ and $F_{ZZ} < 0$: There is diminishing marginal product of Z_t

1.2 Key Equations

	Standard	Augmented
Production function	$Y_t = AF(K_t, N_t)$	$Y_t = AF(K_t, Z_t N_t)$
Motion of capital	$K_{t+1} = I_t + (1-\delta)K_t$	$K_{t+1} = I_t + (1-\delta)K_t$
Investment	$I_t = sY_t$	$I_t = sY_t$
Capital return	$R_t = AF_K(K_t, N_t)$	$R_t = AF_K(K_t, Z_t N_t)$
Wage	$w_t = AF_N(K_t, N_t)$	$w_t = AZ_t F_N(K_t, Z_t N_t)$

1.3 The Evolution of Key Variables

To analyze the evolution path of key variables, we have to make assumptions on Z_t and N_t:

- The initial state: $N_0 = 1$, $Z_0 = 1$
- How they grow over time: $N_t = (1+n)N_{t-1}$, $Z_t = (1+z)Z_{t-1}$

Then we are able to derive the two variables at each time t(geometric series):

- $N_t = (1+n)^t$
- $Z_t = (1+z)^t$

2 Model Transformation

It is useful for us to transform the model in terms of "per efficiency unit of labor terms". We just simply divide key variables by $Z_t N_t$ (recall that we divide the key variables by N_t to derive the variable per worker).
Trick: Divide the key variables with the variables that are changing over time :)

2.1 A Streamline Representation

Capital: Let's derive the capital per efficiency worker step by step:

$$K_{t+1} = sAF(K_t, Z_t N_t) + (1-\delta)K_t$$

Divide both sides by Z_tN_t:

$$\frac{K_{t+1}}{Z_tN_t} = \frac{sAF(K_t, Z_tN_t)}{Z_tN_t} + \frac{(1-\delta)K_t}{Z_tN_t}$$

Apply the CRS/homogeneity of degree 1, the production function on the RHS becomes:

$$\frac{F(K_t, Z_tN_t)}{Z_tN_t} = F(\frac{K_t}{Z_tN_t}, \frac{Z_tN_t}{Z_tN_t}) = F(\hat{k}_t, 1) = f(\hat{k}_t)$$

Then we are able to write the law of motion as:

$$\frac{K_{t+1}}{Z_tN_t} = sAf(\hat{k}_t) + (1-\delta)\hat{k}_t$$

Let's deal with the LHS. Recall our assumption of the evolution of Z_t and N_t, $N_t = (1+n)N_{t-1}$, $Z_t = (1+z)Z_{t-1}$:

$$\frac{K_{t+1}}{Z_tN_t} = \underbrace{\frac{K_{t+1}}{Z_{t+1}N_{t+1}}}_{=\hat{k}_{t+1}} \underbrace{\frac{Z_{t+1}N_{t+1}}{Z_tN_t}}_{=(1+z)(1+n)}$$

, this is a useful trick to deal with the intertemporal variables. Combine the two equations above:

$$\hat{k}_{t+1} = \frac{1}{(1+z)(1+n)}\left[sAf(\hat{k}_t) + (1-\delta)\hat{k}_t\right]$$

Output:

$$Y_t = AF(K_t, Z_tN_t)$$

Divide both sides by Z_tN_t:

$$\hat{y}_t = \frac{Y_t}{Z_tN_t} = \frac{AF(K_t, Z_tN_t)}{Z_tN_t} = Af(\hat{k}_t)$$

Investment:

$$I_t = sY_t$$

Divide both sides by Z_tN_t:

$$\hat{i}_t = \frac{I_t}{Z_tN_t} = \frac{sY_t}{Z_tN_t} = s\hat{y}_t = sAf(\hat{k}_t)$$

Consumption:

$$C_t = (1-s)Y_t$$

Similar to the derivation of investment:

$$\hat{c}_t = (1-s)Af(\hat{k}_t)$$

Capital Return:

$$R_t = AF_K(K_t, Z_tN_t)$$

Recall Euler Theorem, Functions that are homogeneous of degree ρ satisfies:

$$F_K(\gamma K_t, \gamma Z_tN_t) = \gamma^{\rho-1}F_K(K_t, Z_tN_t)$$

Let $\gamma = \frac{1}{Z_t N_t}$, $\rho = 1$:

$$F_K(\frac{K_t}{Z_t N_t}, \frac{Z_t N_t}{Z_t N_t}) = (\frac{1}{Z_t N_t})^{1-1} F_K(K_t, Z_t N_t)$$

$$F_K(K_t, Z_t N_t) = F_K(\hat{k}_t, 1) = f'(\hat{k}_t)$$

Plug in the RHS of $R_t = AF_K(K_t, Z_t N_t)$:

$$R_t = Af'(\hat{k}_t)$$

Wage:

$$w_t = AZ_t F_N(K_t, Z_t N_t)$$

Let $\rho = 1$ and apply Euler Theorem:

$$AF(K_t, Z_t N_t) = F_K(K_t, Z_t N_t)K_t + F_N(K_t, Z_t N_t)Z_t N_t$$

Divide both sides by $Z_t N_t$, and apply the homogeneity of production function:

$$\frac{F(K_t, Z_t N_t)}{Z_t N_t} = \frac{F_K(K_t, Z_t N_t)K_t}{Z_t N_t} + \frac{F_N(K_t, Z_t N_t)Z_t N_t}{Z_t N_t}$$

$$F(\frac{K_t}{Z_t N_t}, \frac{Z_t N_t}{Z_t N_t}) = F_K(K_t, Z_t N_t)\frac{K_t}{Z_t N_t} + F_N(K_t, Z_t N_t)\frac{Z_t N_t}{Z_t N_t}$$

Recall:

$$F_K(K_t, Z_t N_t) = F_K(\hat{k}_t, 1) = f'(\hat{k}_t)$$

Rearrange:

$$f(\hat{k}_t) = f'(\hat{k}_t)\hat{k}_t + F_N(K_t, Z_t N_t)$$

$$F_N(K_t, Z_t N_t) = f(\hat{k}_t) - f'(\hat{k}_t)\hat{k}_t$$

Plug in the RHS of $w_t = AZ_t F_N(K_t, Z_t N_t$:

$$w_t = AZ_t(f(\hat{k}_t) - f'(\hat{k}_t)\hat{k}_t)$$

3 Steady State

3.1 Big Picture

Steps of analyzing a shock to the economy:

- Step 1: Solve the key equations
- Step 2: Solve the steady state
- Step 3: Analyze the new steady state
- Step 4: Find the transition path

3.2 Example: An increase in A

3.2.1 Step 1: Solve the key equations in the augmented Solow growth model

First, write down key equations at the aggregate level:

- $Y_t = AF(K_t, Z_t N_t)$
- $K_{t+1} = I_t + (1-\delta)K_t$
- $I_t = sY_t$
- $C_t = (1-s)Y_t$
- $R_t = AF_K(K_t, Z_t N_t)$
- $w_t = AZ_t F_N(K_t, Z_t N_t)$

Second, represent the key equations at the "per effective worker" level:

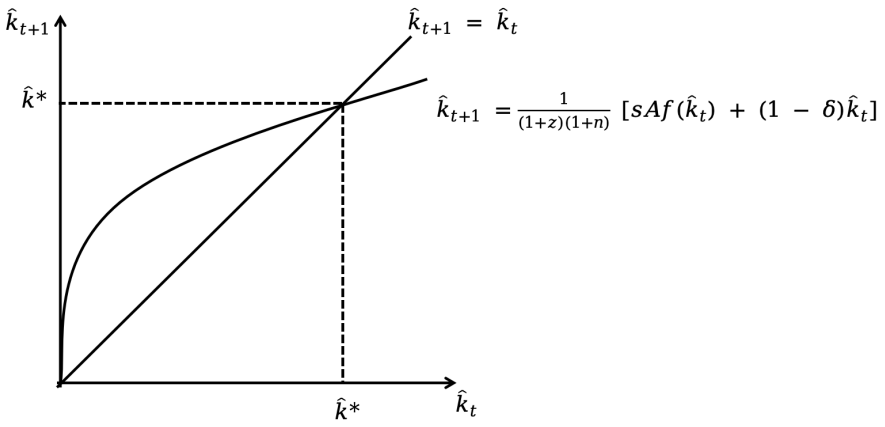
- $\hat{y}_t = Af(\hat{k}_t)$
- $\hat{k}_{t+1} = \frac{1}{(1+z)(1+n)}\left[sAf(\hat{k}_t) + (1-\delta)\hat{k}_t\right]$
- $\hat{i}_t = sAf(\hat{k}_t)$
- $\hat{c}_t = (1-s)Af(\hat{k}_t)$
- $R_t = Af'(\hat{k}_t)$
- $w_t = AZ_t(f(\hat{k}_t) - f'(\hat{k}_t)\hat{k}_t)$

3.2.2 Step 2: Solve the steady state

If the production function is given(i.e. Cobb-Douglas production function), we are able to solve all the variables above. Just follow the steps:

- Let $\hat{k}_{t+1} = \hat{k}_t$, then we are able to solve the capital per effective worker $\hat{k}^*$at the steady state by solving:
$$\hat{k}^* = \frac{1}{(1+z)(1+n)}\left[sAf(\hat{k}^*) + (1-\delta)\hat{k}^*\right]$$
- Given $\hat{k}^*$, we are able to solve $\hat{y}^*$, $\hat{i}^*$, $\hat{R}^*$. They are constant at the steady state.
- Wage is a special case, it depends on the level of effciency Z_t at the steady state. We are not able to solve the steady-state wage unless we know the level of efficiency at the steady state.

The graph is similar to the previous one:



Unlike the steady state of the standard case, now there is a growth at the steady state. We analyze the growth at the aggregate level. Take the growth rate of capital as an example:

- Aggregate growth:

$$\frac{K_{t+1}}{K_t} = \frac{\hat{k}_{t+1} Z_{t+1} N_{t+1}}{\hat{k}_t Z_t N_t}$$

, at the steady state $\hat{k}_{t+1} = \hat{k}_t$. Then,

$$\frac{K_{t+1}}{K_t} = \frac{Z_{t+1} N_{t+1}}{Z_t N_t} = (1+z)(1+n)$$

,

$$g_K \approx z + n$$

- Growth of capital per worker:

$$\frac{K_{t+1}/N_{t+1}}{K_t/N_t} = \frac{\hat{k}_{t+1} Z_{t+1}}{\hat{k}_t Z_t} = 1 + z$$

,

$$g_k = z$$

Following the same method, we are able to derive that $g_K = g_Y = g_C = g_I = z + n$, $g_k = g_y = g_c = g_i = z$.
It is a little different when it comes to the capital return R and wage w:

- $R_t = Af'(\hat{k}_t)$: At steady state, capital return remain constant since it is a function of $\hat{k}^*$. So, the growth of capital return is 0.
- $w_t = AZ_t(f(\hat{k}_t) - f'(\hat{k}_t)\hat{k}_t)$: $g_w = \frac{w_{t+1}}{w_t} = \frac{Z_{t+1}}{Z_t} = z$

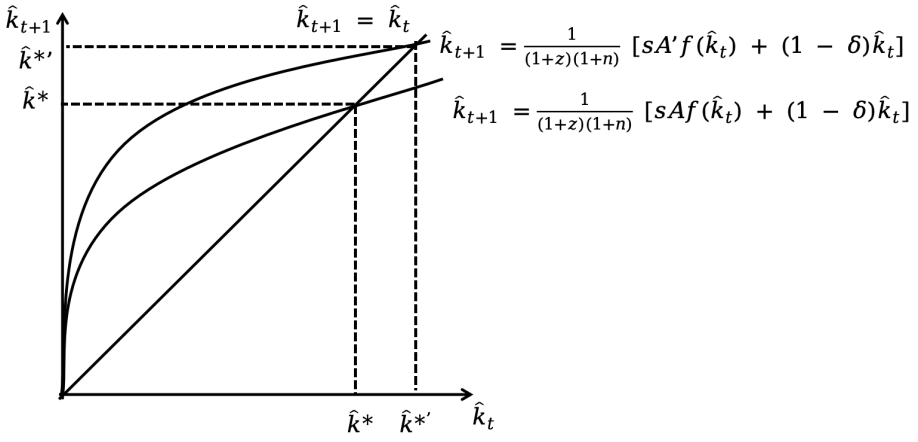
3.2.3 Step 3: Analyze the new steady state

If there is a sudden jump in Total Factor Productivity (TFP) A, how will the new steady state change? To answer this questions, we directly use the equations at steady state.

We start with the capital. Capital per effective worker at steady state is given by the solution of:

$$\hat{k}^* = \frac{1}{(1+z)(1+n)}\left[sAf(\hat{k}^*) + (1-\delta)\hat{k}^*\right]$$

, that is, the capital per effective worker increases when there is an increase in A. See the graph below:



It may not be obvious in terms of this general form, but the answer will be clear if we know the production function f. For the Cobb-Douglas production function, $f(\hat{k}^*) = (\hat{k}^*)^\alpha$, we are able to solve $\hat{k}^*$ explicitly:

$$\hat{k}^* = \left(\frac{sA}{z+n+nz+\delta}\right)^{\frac{1}{1-\alpha}}$$

. It's obvious that $A \uparrow \Rightarrow \hat{k}^* \uparrow$.

Key variables in terms of "per effective worker" at new steady state:
Given $A \uparrow$ and $\hat{k}^* \uparrow$, the level of the key variables at new steady state is given by:

- $\hat{y}^* = Af(\hat{k}^*)$, $f' > 0$. Since A increases and $\hat{k}^*$ at new steady state is greater than that of the old one, $\hat{y}^*$ also increases.
- $\hat{i}^* = sAf(\hat{k}^*)$, $f' > 0$. s remains unchanged, A increases and $\hat{k}^*$ increases, so $\hat{i}^*$ also increases.
- $\hat{c}^* = (1-s)Af(\hat{k}^*)$, $f' > 0$. $1-s$ remains unchanged, A increases and $\hat{k}^*$ increases, so $\hat{c}^*$ also increases.
- $R^* = Af'(\hat{k}^*)$. Since $\hat{k}^*$ increases, $f'' < 0$, we know $f'(\hat{k}^*)$ decreases. However, since TFP A increases, the capital return R^* at new steady state is ambiguous. It varies case by case.
- $w_t^* = AZ_t(f(\hat{k}^*) - f'(\hat{k}^*)\hat{k}^*)$. Wage depends on the efficiency, so we are not able to solve the wage at the new steady state. We only know $g_w = z$.

Growth rate of key variables in terms of "per worker" at new steady state The growth rate of the key variables in terms of "per worker" is:

- $g_k = g_y = g_c = g_i = z$
- $g_r = 0$, $g_w = z$

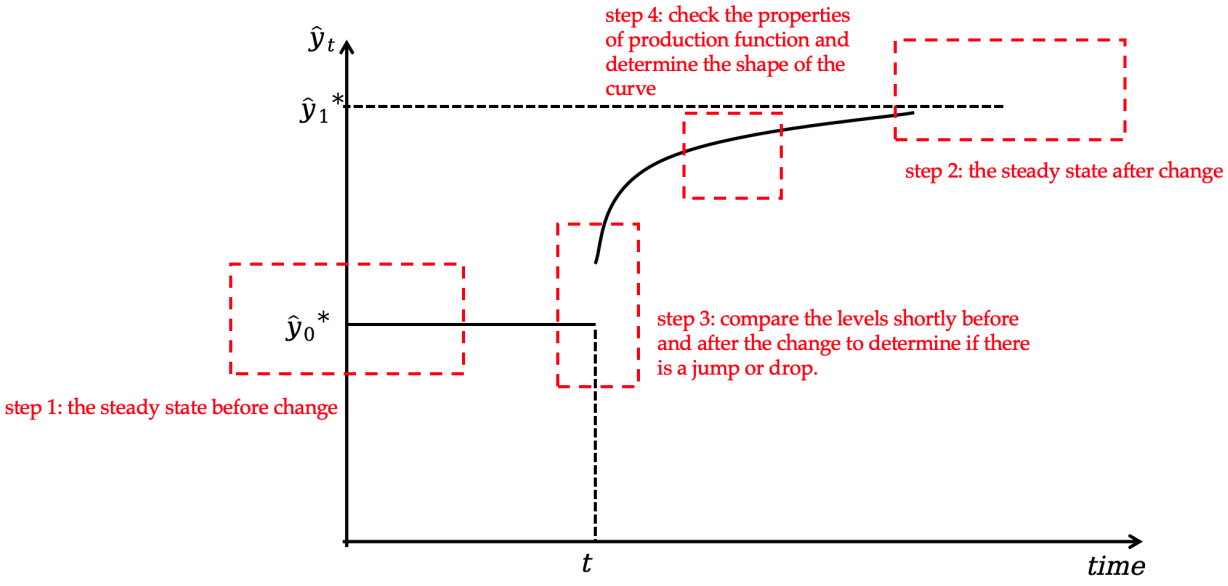
The growth rate of each variable is uncorrelated with TFP A or capital at any steady state. So, the growth rate(the slope) doesn't change at new steady state.

3.2.4 Step 4: Find the transition path

Now we know the level and growth rate of key variables before and after the change. Then how does the economy converge to the new steady state after the sudden increase of TFP A? Is there a jump or drop as soon as the change occurs? How is the shape of the transition path like? Is it convex or concave?

Key variables in terms of "per effective worker"

- $\hat{k}$: There is no jump since capital is accumulated gradually. See Handout 3 for the detailed proof.
  What if the population N suddenly changes?
- $\hat{y}$: $\hat{y} = Af(\hat{k})$. As soon as the change occurs, the capital per effective worker doesn't change. But there is an sudden increase in A, which means at the $\hat{k}$, the output per effective worker $\hat{y}'$ is strictly greater than the previous output per effective worker $\hat{y}$. So, there is a jump of $\hat{y}$ as soon as the change occurs.
- $\hat{c}$: $\hat{c} = (1-s)\hat{y}$. Consumption per effective worker is just part of the output per effective worker. So, there is also a jump as soon as the change occurs.
- $\hat{i}$: $\hat{i} = s\hat{y}$. There is a jump. The reason is similar.
- R: $R = Af'(\hat{k})$. As soon as the change occurs, the capital per effective worker doesn't change, so $f'(\hat{k})$ doesn't change. Given the sudden increase in A, at $\hat{k}_t$(t is the time when the change occurs), the capital return $R' = A'f(\hat{k}_t)$ is strictly greater than $R = Af(\hat{k}_t)$. So, there is a jump in R as soon as A increases suddenly. After that, as $\hat{k}$ accumulates gradually, f' falls gradually since $f'' < 0$. So after the jump, R falls gradually and converge to the new steady state.



Key variables in terms of "per worker": growth rate

We've already proved that the growth rate of the key variables doesn't change at the new steady state. That is, the slope of the curve doesn't change. But what happens as soon as the change occurs?

A quick detour: Why we plot $\ln x_t$ against time t? Variable x_t grows at a constant growth rate z, then given the initial state x_0, x_t for $\forall t$ is given by:

$$x_t = x_0(1+z)^t$$

If we plot x_t against time t, this will be an exponential function. However, if we take log at both sides:

$$\ln x_t = \ln x_0 + t\ln(1+z)$$

So, $\ln x_t$ is linear in time t, which is better to show that the growth rate is constant at steady state.

Is there a jump or drop? Take capital per worker k_t as an example:

$$\ln k_t = \ln(\hat{k}_t Z_t)$$

, since there is no jump in $\hat{k}_t$, there is also no jump in $\ln k_t$ (Z_t approximately doesn't change in a very short time). Similarly, since there is a jump in $\hat{y}_t$, $\hat{c}_t$, $\hat{i}_t$, there is also a jump in terms of the log form.

Why the curve is concave between the steady states? Intuitively, because the second order derivative of the production function is negative, $f'' < 0$. Therefore, the accumulation of capital per worker is slower and slower. So, the "growth rate" of the growth rate becomes smaller and smaller. That's why the curve is flattening between the steady states.

Wage: $w_t^* = AZ_t(f(\hat{k}^*) - f'(\hat{k}^*)\hat{k}^*)$. That is, $\frac{w_{t+1}}{w_t} = 1+z$, $g_w = z$. The growth rate of wage is constant at the old and new steady states. The slope of the curve $\ln w_t$ against time t doesn't

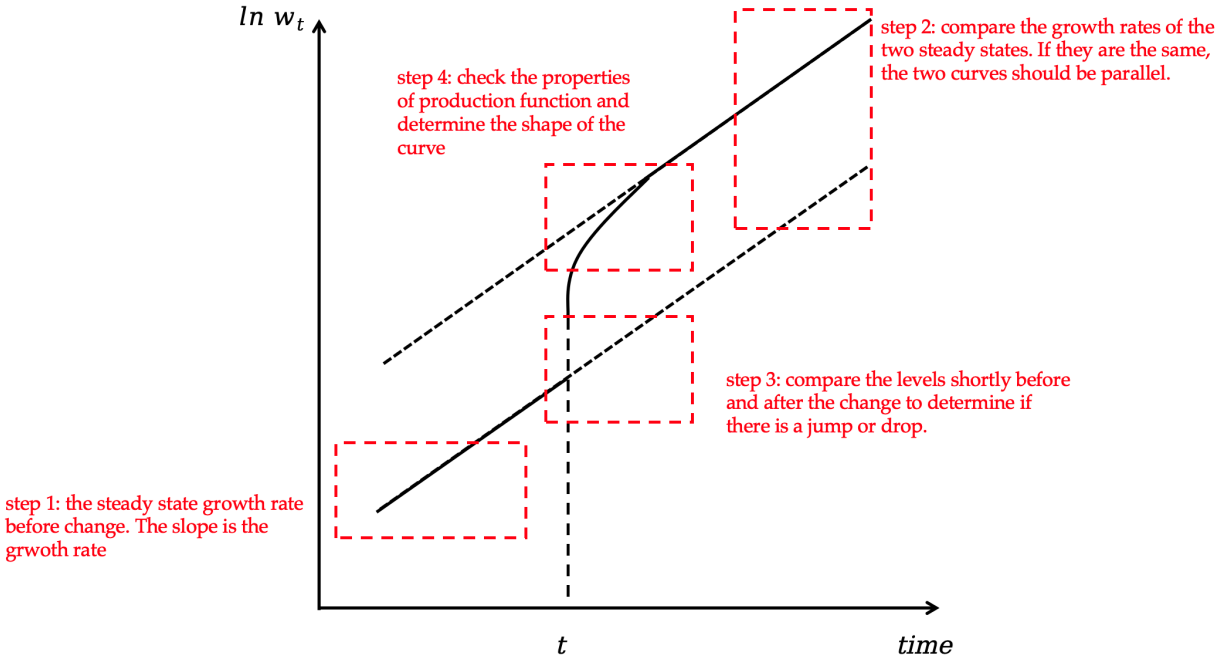
change after the economy reach the new steady state. As soon as the TFP jumps from A to $A' > A$, $\ln w_t$ also jumps to a higher level since $\ln w_t$ is a function of A. Let Δt represent a very short time, where we can treat the efficiency Z_t as unchanged, $Z_{t+\Delta t} \approx Z_t$:

$$w_{t+\Delta t} = A' Z_{t+\Delta t}(f(\hat{k}^*) - f'(\hat{k}^*)\hat{k}^*)$$

$$w_t = A Z_t(f(\hat{k}^*) - f'(\hat{k}^*)\hat{k}^*)$$

$$w_{t+\Delta t} > w_t$$

, so there is a jump.



4 Hints on HW4

- Capital is accumulated gradually, but the accumulation of capital per worker also depends on the population.
- Implicit differentiation: Take the derivative of both sides of the equation