1 Optimal Consumption and Saving

1.1 Utility function

Household's utility function:

$$U = \underbrace{u(C_t)}_{\text{Utility today}} + \underbrace{\beta u(C_{t+1})}_{\text{Discounted utility tomorrow}}$$

Key properties of utility function:

- $\beta < 1$: Discount factor. Intuition: Household's value utility today more.
- $u'(c) > 0$: $c \uparrow \Rightarrow u(c)$. The more consumption people get, the happier they are
- $u''(c) < 0 : \ c \uparrow \Rightarrow u'(c) \downarrow$. Diminishing marginal utility. The additional utility by one more unit of consumption decreases as the consumption goes up.

1.2 Household's problem

$$\max_{C_t, C_{t+1}} u(C_t) + \beta u(C_{t+1})$$

s.t.

$$C_t + \underbrace{S_t}_{\text{Savings today}} = Y_t$$

$$C_{t+1} = Y_{t+1} + \underbrace{(1+r_t)S_t}_{\text{Return on savings}}$$

Substitute S_t and rearrange the two-period budget constraints above, the single constraint is given by:

$$\underbrace{C_t}_{\text{Consumption today}} + \underbrace{\frac{C_{t+1}}{1+r_t}}_{\text{PV of consumption tomorrow}} = \underbrace{Y_t}_{\text{Income today}} + \underbrace{\frac{Y_{t+1}}{1+r_t}}_{\text{PV of the income tomorrow}}$$

Solve the household's problem, the Euler equation is given by:

$$u'(C_t) = \beta(1+r_t)u'(C_{t+1})$$

Intuition: Euler equation implies how much consumption today household is willing to give up to exchange for a particular unit of consumption tomorrow:

- LHS: Household gives up 1 unit of goods for consumption, which means it gives up $u'(c_t)$ units of utility. Household then save 1 unit of goods, which will bring $1 + r_t$ units of goods in return for period $t + 1$.
- RHS: By saving 1 unit of goods, household receive $1 + r_t$ units of goods for period $t + 1$, which brings $(1 + r_t)u'(C_{t+1})$ units of utility for next period. The discounted utility is how much household value the utility tomorrow at time t, $\beta(1 + r_t)u'(C_{t+1})$
- The utility maximization: LHS equals RHS.

- If $u'(C_t) > \beta(1+r_t)u'(C_{t+1})$, the marginal utility from consuming 1 unit of goods today is greater than the PV(present value) of the uility by consuming $1 + r_t$ units goods tomorrow. Then household chooses to consume the goods today, then $C_t \uparrow$. Since $u''(C_t) < 0$, when $C_t \uparrow$, $u'(C_t) \downarrow$, the inequality will go back to the equality.
- If $u'(C_t) < \beta(1+r_t)u'(C_{t+1})$, $C_{t+1} \uparrow$ for the same reason. Then $u'(C_{t+1}) \downarrow$, the inequality will go back to the equality.

We can also interpret Euler equation as marginal rate of substitution equals the relative price:

$$\frac{u'(C_t)}{\beta u'(C_{t+1})} = 1 + r_t$$

Intuition:

- Indifference curve:

$$MRS_{C_t,C_{t+1}} = -\frac{d_{C_{t+1}}}{d_{C_t}}$$

  - LHS: Marginal rate of substitution:

$$MRS_{C_t,C_{t+1}} = \frac{MU_{C_t}}{MU_{C_{t+1}}} = \frac{u'(C_t)}{\beta u'(C_{t+1})}$$

    Household's problem is utility maximization. We can take the "utility" as the "profit" in the firm problem. 1 unit of goods "produce" $u'(C_t)$ units of utility today and $\beta u'(C_{t+1})$ units of utility tomorrow(PV). So, the "benefit" of consuming 1 unit of goods today relative to consuming it tomorrow is, $\frac{u'(C_t)}{\beta u'(C_{t+1})}$

  - RHS: 1 unit of goods today can be exchanged for $-\frac{d_{C_{t+1}}}{d_{C_t}}$ units of goods tomorrow. See the constraint:

$$C_t + \frac{C_{t+1}}{1+r_t} = Y_t + \frac{Y_{t+1}}{1+r_t}$$

    Take first order derivative w.r.t. C_t on both sides:

$$-\frac{d_{C_{t+1}}}{d_{C_t}} = 1 + r_t$$

    So, the "cost" of consuming 1 unit of goods is $1 + r_t$ units of goods tomorrow

1.3 Case analysis

Key points of graphic analysis:

- Indifference curve: $U = u(C_t) + \beta u(C_{t+1})$
- Budget line: $C_t + \frac{1}{1+r_t}C_{t+1} = Y_t + \frac{1}{1+r_t}Y_{t+1}$. The slope is $-(1 + r_t)$. Note that (Y_t, Y_{t+1}) is always a point on the budget line.
- Tangent point: Satisfies Euler equation: $\frac{u'(C_t)}{\beta u'(C_{t+1})} = 1 + r_t$

We want to examine the change in current consumption when the exogeneous variables change. To analyze these questions, we employ the first order condition from household's utility maximization problem:

$$u'(C_t) - (1 + r_t)\beta u'\left((1 + r_t)(Y_t - C_t) + Y_{t+1}\right) = 0$$

1.3.1 Case I: Y_t: Shifts the budget line

Take first order derivatives w.r.t. Y_t on both sides:

$$u''(C_t)\frac{\partial C_t}{\partial Y_t} + (1+r_t)^2\beta u''(C_{t+1})\frac{\partial C_t}{\partial Y_t} - (1+r_t)^2\beta u''(C_{t+1}) = 0$$

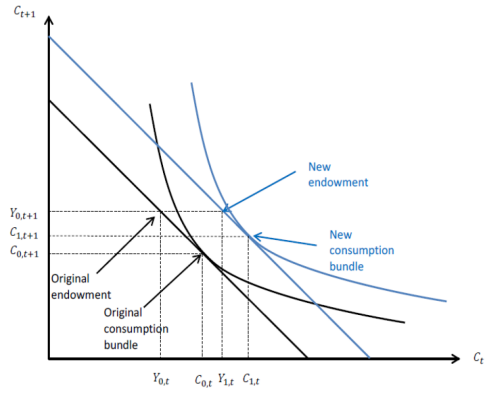
$$\frac{\partial C_t}{\partial Y_t} = \frac{(1+r_t)^2\beta u''(C_{t+1})}{u''(C_t) + (1+r_t)^2\beta u''(C_{t+1})}$$

To determine the sign of $\frac{\partial C_t}{\partial Y_t}$, notice that $u''(c) < 0$, so both the bottom and the top are negative. Then, the fraction is positive. Also, notice that absolute value of the bottom is greater than that of the top, $(1+r_t)^2\beta u''(C_{t+1}) < u''(C_t) + (1+r_t)^2\beta u''(C_{t+1})$. That is, the fraction is less than 1:

$$0 < \frac{\partial C_t}{\partial Y_t} < 1$$

Intuition: An increase in income today won't fully increase consumption today because it raise the consumption in two periods simultaneously. Since the cost of consuming 1 unit of goods today is $1 + r_t$ units goods tomorrow. If it fully increase the consumption today, then the Euler equation will be violated, which means the household is no longer maximize their utility.

Figure 9.5: Increase in Y_t



1.3.2 Case II: Y_{t+1}: Shifts the budget line

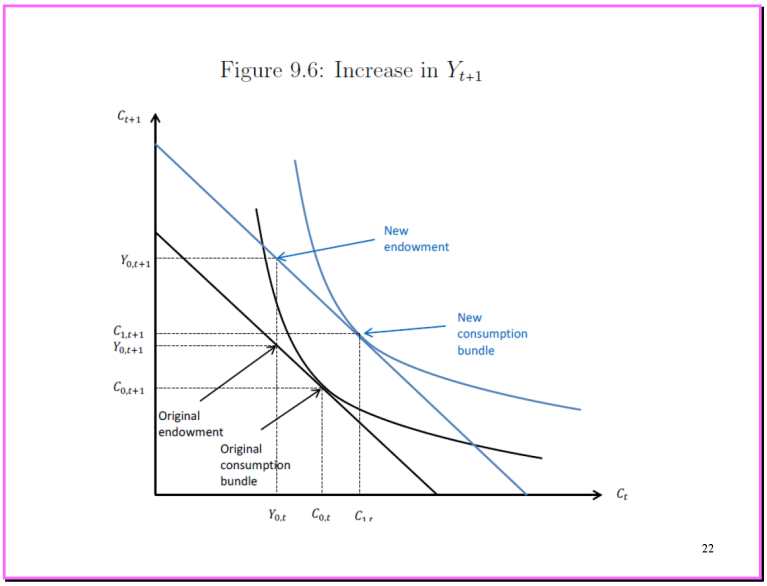
Take first order derivatives w.r.t. Y_{t+1} on both sides:

$$u''(C_t)\frac{\partial C_t}{\partial Y_{t+1}} + (1+r_t)^2\beta u''(C_{t+1})\frac{\partial C_t}{\partial Y_{t+1}} - (1+r_t)\beta u''(C_{t+1}) = 0$$

$$\frac{\partial C_t}{\partial Y_{t+1}} = \frac{(1+r_t)\beta u''(C_{t+1})}{u''(C_t) + (1+r_t)^2\beta u''(C_{t+1})} > 0$$

The sign is positive for the similar reason to the previous one. Note that we don't know the relationship between $\frac{\partial C_t}{\partial Y_{t+1}}$ and 1, since when the future income rises enough, consumers will borrow

to consume more at time t. This is because the same amount of utility worth more at time t than that at time $t+1$ because of the discount factor β.



1.3.3 Case III: r_t: Rotates the budget line around point (Y_t, Y_{t+1})

Take first order derivatives w.r.t. r_t on both sides

$$u''(C_t)\frac{\partial C_t}{\partial r_t} + (1+r_t)^2\beta u''(C_{t+1})\frac{\partial C_t}{\partial r_t} - \beta u'(C_{t+1}) - (1+r_t)\beta(Y_t - C_t)u''(C_{t+1}) = 0.$$

$$\frac{\partial C_t}{\partial r_t} = \frac{\beta u'(C_{t+1}) + (1+r_t)\beta(Y_t - C_t)u''(C_{t+1})}{u''(C_t) + (1+r_t)^2\beta u''(C_{t+1})}$$

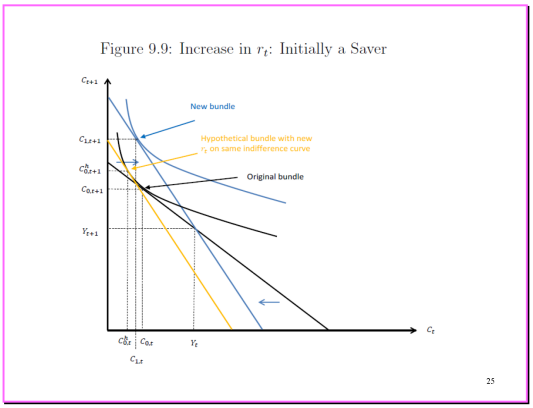
$$\frac{\partial C_t}{\partial r_t} = \underbrace{\frac{\beta u'(C_{t+1})}{u''(C_t) + (1+r_t)^2\beta u''(C_{t+1})}}_{\text{Substitution effect}} + \underbrace{\frac{(1+r_t)\beta(Y_t - C_t)u''(C_{t+1})}{u''(C_t) + (1+r_t)^2\beta u''(C_{t+1})}}_{\text{Income effect}}$$

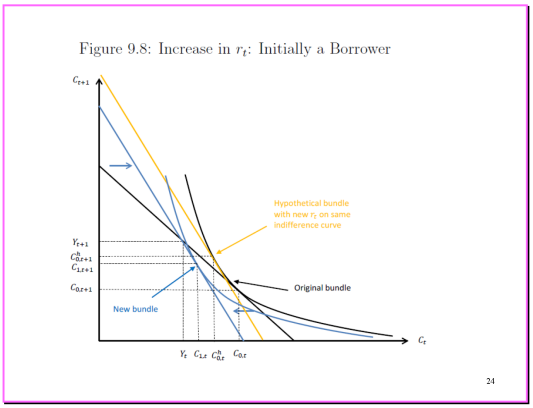
Substitution effect: $\left.\frac{\partial C_t}{\partial r_t}\right|_{\text{Substitution}} < 0$. $1+r_t$ is the "cost" of consumption today in terms of consumption tomorrow. When $r_t \uparrow$, it is costly to consume today. Instead of consuming, if household save and consume at time $t+1$, the same amount of goods will bring more utility than that before the increase in r_t. So the substitution effect implies the increase in r_t will result in a decrease in C_t.

Income effect:

- $\left.\frac{\partial C_t}{\partial r_t}\right|_{\text{Income}} > 0$ if $Y_t - C_t > 0$. If household is saver, an increase in r_t is equivalent to an increase in the future income.
- $\left.\frac{\partial C_t}{\partial r_t}\right|_{\text{Income}} < 0$ if $Y_t - C_t < 0$. If household is borrower, an increase in r_t is equivalent to an decrease in the future income.

We will assume overall the substitution effect dominates.





2 Exercise [1]

Let's assume the household maximizes a lifetime utility function

$$U = u(C_t) + \beta u(C_{t+1})$$

such that

$$u(C_t) = \frac{C_t^{1-\sigma} - 1}{1-\sigma}, \quad \sigma > 0$$

Assume all features in a standard two-period model introduced in lecture hold. Additionally, assume $\beta(1 + r_t) = 1$.

1. Derive the Euler equation of the household. What does the assumption $\beta(1 + r_t) = 1$ imply? What if $\beta(1 + r_t) > 1$?

2. Express consumptions for both periods (C_t and C_{t+1}) as a function of two periods' incomes and interest rate.

3. Provide the conditions for Y_t, Y_{t+1}, r_t under which the household becomes a saver or a borrower.

[1] This exercise is based on previous exercise by Junhyong Kim and Saerang Son

3 Appendix & Solutions

3.1 Mankiw, Romer and Weil (MRW) model

Decompose the aggregate variable in log form is a useful way to conduct the empirical analysis. MRW is an extension of the augmented Solow model. By adding the human capital, the production becomes:

$$Y_t = AK_t^{\alpha} H_t^{\beta} (Z_t N_t)^{1-\alpha-\beta}$$

Similar to the way we derive the "capital per effective worker", we have the two variables in terms of "per effective worker":

$$\hat{k}_{t+1} = \frac{1}{(1+z)(1+n)} \left(s_k \hat{y}_t + (1-\delta)\hat{k}_t \right)$$

$$\hat{h}_{t+1} = \frac{1}{(1+z)(1+n)} \left(s_h \hat{y}_t + (1-\delta)\hat{h}_t \right)$$

Then the steady state takes the similar form to the augmented Solow model:

$$\hat{k}^* = \frac{1}{(1+z)(1+n)} \left(s_k A \hat{k}^{*\alpha} \hat{h}^{*\beta} + (1-\delta)\hat{k}^* \right)$$

$$\hat{h}^* = \frac{1}{(1+z)(1+n)} \left(s_h A \hat{k}^{*\alpha} \hat{h}^{*\beta} + (1-\delta)\hat{h}^* \right)$$

The key conclusion here is:

$$\frac{\hat{k}^*}{\hat{h}^*} = \frac{s_k}{s_h}$$

3.2 Optimal Consumption and Saving: Household's problem

3.2.1 Single constraint

Two-period budget constraint is given by:

$$C_t + S_t = Y_t$$

$$C_{t+1} = Y_{t+1} + (1+r_t)S_t$$

For $C_{t+1} = Y_{t+1} + (1+r_t)S_t$, divide both sides by $1+r_t$, we have:

$$C_t + S_t = Y_t$$

$$\frac{1}{1+r_t} C_{t+1} = \frac{1}{1+r_t} Y_{t+1} + S_t$$

Combine these two equations:

$$C_t + S_t + \frac{1}{1+r_t} C_{t+1} = Y_t + \frac{1}{1+r_t} Y_{t+1} + S_t$$

Substract S_t from both sides:

$$C_t + \frac{1}{1+r_t} C_{t+1} = Y_t + \frac{1}{1+r_t} Y_{t+1}$$

3.2.2 Solve the household's maximization problem

$$\max_{C_t, C_{t+1}} u(C_t) + \beta u(C_{t+1})$$

s.t.

$$C_t + S_t = Y_t$$
$$C_{t+1} = Y_{t+1} + (1 + r_t)S_t$$

Solution 1: Solve C_{t+1} from the single constraint:

$$C_t + \frac{1}{1+r_t} C_{t+1} = Y_t + \frac{1}{1+r_t} Y_{t+1}$$
$$C_{t+1} = \underbrace{(1+r_t)(Y_t - C_t)}_{\text{Return from savings}} + \underbrace{Y_{t+1}}_{\text{New ouput}}$$

Substitute into the utility function and get an unconstrained maximization problem:

$$\max_{C_t} U = u(C_t) + \beta u\left((1+r_t)(Y_t - C_t) + Y_{t+1}\right)$$

Take first order derivative w.r.t C_t. The maximization problem requires FOC to be 0:

$$\frac{\partial U}{\partial C_t} = u'(C_t) - (1+r_t)\beta u'(\underbrace{(1+r_t)(Y_t - C_t) + Y_{t+1}}_{C_{t+1}}) = 0$$

Rearrange and get the Euler equation:

$$u'(C_t) = \beta(1+r_t)u'(C_{t+1})$$

Solution 2: Set the Lagrangian:

$$\mathcal{L} = u(C_t) + \beta u(C_{t+1}) - \lambda_t(C_t + S_t - Y_t) - \lambda_{t+1}(C_{t+1} - Y_{t+1} - (1+r_t)S_t)$$

FOC:

$$[C_t] : u'(C_t) = \lambda_t$$
$$[C_{t+1}] : \beta u'(C_{t+1}) = \lambda_{t+1}$$
$$[S_t] : \lambda_t = (1+r_t)\lambda_{t+1}$$

Combine the equations above, we have:

$$u'(C_t) = \beta(1+r_t)u'(C_{t+1})$$

3.3 Exercise Solution

Let's assume the household maximizes a lifetime utility function

$$U = u(C_t) + \beta u(C_{t+1})$$

such that

$$u(C_t) = \frac{C_t^{1-\sigma} - 1}{1-\sigma}, \quad \sigma > 0$$

Assume all features in a standard two-period model introduced in lecture hold. Additionally, assume $\beta(1+r_t) = 1$.

1. Derive the Euler equation of the household. What does the assumption $\beta(1+r_t)=1$ imply? What if $\beta(1+r_t)>1$?

   Answer: The first order condition with respect to C_t is

   $$C_t^{-\sigma} = C_{t+1}^{-\sigma}\beta(1+r_t)$$

   $$\left(\frac{C_{t+1}}{C_t}\right)^{\sigma} = \beta(1+r_t)$$

   $\beta(1+r_t)=1$ means that $C_t = C_{t+1}$. If $\beta(1+r_t)>1$, then $C_t < C_{t+1}$.

2. Express consumptions for both periods (C_t and C_{t+1}) as a function of two periods' incomes and interest rate.

   Answer: From the first order condition, we know that $C_t = C_{t+1}$. This means

   $$Y_t - S_t = Y_{t+1} + (1+r_t)S_t$$

   The saving as a function of output and interest rate is

   $$S_t = \frac{Y_t - Y_{t+1}}{2+r_t}$$

   The consumptions for both periods are

   $$C_t = C_{t+1} = Y_t - S_t = \frac{(1+r_t)Y_t + Y_{t+1}}{2+r_t}$$

3. Provide the conditions for Y_t, Y_{t+1}, r_t under which the household becomes a saver or a borrower. Answer: From Question 2, the saving as a function of output and interest rate is

   $$S_t = \frac{Y_t - Y_{t+1}}{2+r_t}$$

   The household becomes a saver if $Y_t > Y_{t+1}$ and becomes a borrower if $Y_t < Y_{t+1}$.