1 Model with Uncertainty

To simplify the analysis, we make the following assumptions:

- $\beta = 1$: No discount factor
- $r_t = 0$: Interest rate equals 0

So household's maximization problem becomes:

$$\max_{C_t, C_{t+1}} u(C_t) + u(C_{t+1})$$

s.t.

$$C_t + C_{t+1} = Y_t + Y_{t+1}$$

Denote $u(C_t) = \ln(C_t)$, the optimal consumption without uncertainty is:

$$C_t = \frac{1}{2}(Y_t + Y_{t+1})$$
$$C_{t+1} = \frac{1}{2}(Y_t + Y_{t+1})$$

1.1 Expectation of future endowment

Household makes decision based on the expectation of future endowment:

$$\max_{C_t, C_{t+1}} u(C_t) + E_t u(C_{t+1})$$

Timing

- $T = t$: Household "receives" endowment Y_t, and consumes C_t. The future endowment is uncertain to the household. Household only knows that it would be either Y_{t+1}^h or Y_{t+1}^l.
- $T = t + 1$: Y_{t+1} is realized and equals either Y_{t+1}^h or Y_{t+1}^l. Household "receives" Y_{t+1} and consumes C_{t+1}

Since at time t, household only know the expectation of the future income, the optimal decision of consumptions depends on the current endowment and expectation of future endowment.

Given $\mathbb{P}(Y_{t+1} = Y_{t+1}^h) = \frac{1}{2}$ and $\mathbb{P}(Y_{t+1} = Y_{t+1}^l) = \frac{1}{2}$, the maximization problem of household is:

$$\max_{C_t, C_{t+1}} u(C_t) + \underbrace{\frac{1}{2}u(C_{t+1}^h) + \frac{1}{2}u(C_{t+1}^l)}_{E_t u(C_{t+1})}$$

s.t.

$$C_t + S_t = Y_t$$
$$C_{t+1}^h = Y_{t+1}^h + S_t$$
$$C_{t+1}^\ell = Y_{t+1}^\ell + S_t$$

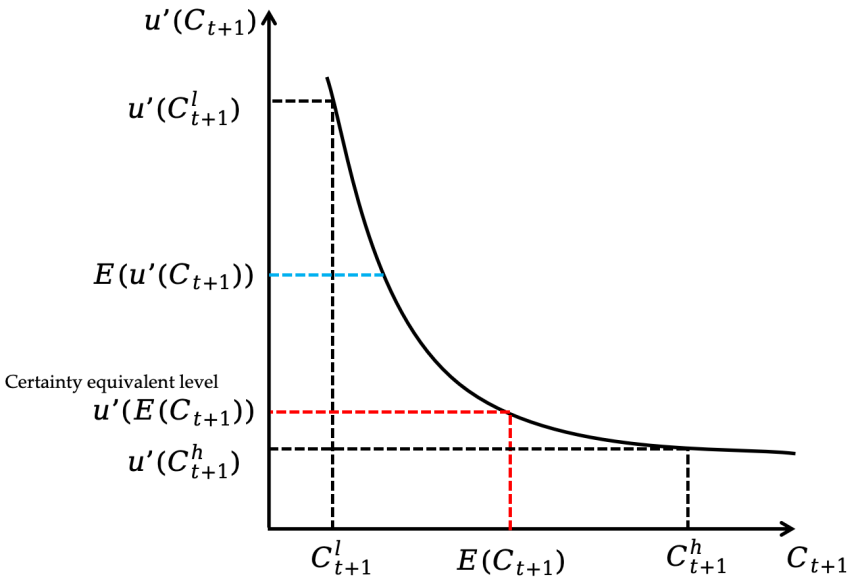
Generally, we can write the Euler equation as:

$$u'(C_t) = E_t(u'(C_{t+1}))$$

1.2 Precautionary savings

Precautionary saving is when your saving under uncertainty is **greater than** the "certainty equivalent" saving rate.
Key condition for whether there is a precautionary savings:

$$\text{Positive precautionary savings} \Leftrightarrow u'''(.) > 0$$



Take-away:

- The expectation of future consumption if $E(C_{t+1})$, which equals $\frac{1}{2}C^h_{t+1} + \frac{1}{2}C^l_{t+1}$
- The certainty equivalent level of marginal utility is $u'(E(C_{t+}))$
- If the endowment is Y^h_{t+1}, then the future consumption is C^h_{t+1}, the corresponding marginal utility is $u'(C^h_{t+1})$; if the endowment is Y^l_{t+1}, then the future consumption is C^l_{t+1}, the corresponding marginal utility is $u'(C^l_{t+1})$. Then the RHS of Euler equation, $E(u'(C_{t+1}))$, is given by $\frac{1}{2}u'(C^h_{t+1}) + \frac{1}{2}u'(C^l_{t+1})$

Since $E(u'(C_{t+1})) > u'(E(C_{t+}))$, given $u'(c)$ is decrease in c, we know that the consumption level is lower than the certainty equivalent consumption level. So, the savings is greater than the certainty equivalent savings. So, there is precautionary saving.

Why $u'''() > 0$: The third order derivative implies the shape of the marginal utility, which indicates the relationship of $E(u'(C_{t+1}))$ and $u'(E(C_{t+}))$.

1.3 Increase in uncertainty

If there is an increase in uncertainty, then the difference between Y^h and Y^l increases.
We employ the Euler equation to find if the precautionary savings increase or decrease:

$$u'(C_t\downarrow)\uparrow = E_t(u'(C_{t+1}))\uparrow$$

Let's do the backward induction:

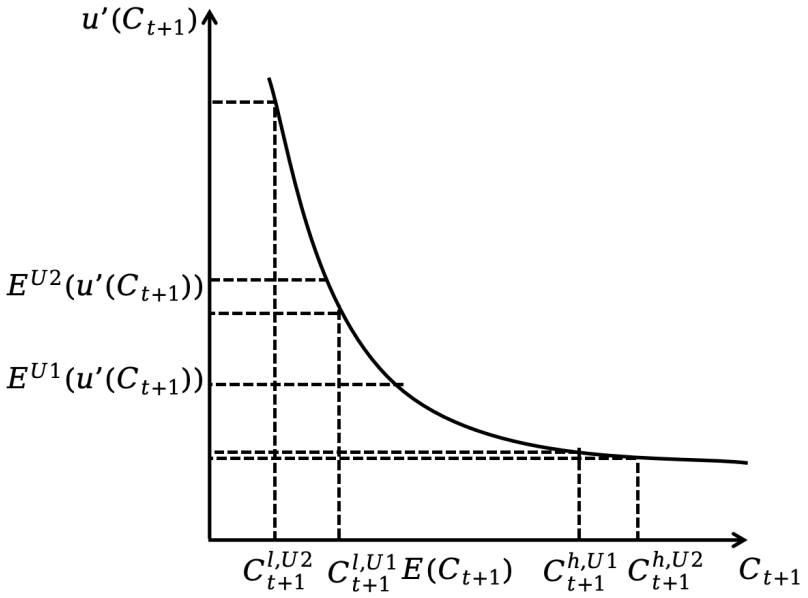
- The endowment Y_t of the first period is fixed. Given the constraint of first period, $C_t+S_t = Y_t$. If C_t increases, then S_t decreases; on the contrast, if C_t decreases, then S_t increases
- Since $u''(C_t) < 0$. If $u'(C_t)$ increases, then C_t decreases; if $u'(C_t)$ decreases, then C_t increases.
- By the Euler equation: $u'(C_t) = E_t(u'(C_{t+1}))$. If the RHS increases, then the LHS increases.

Suppose there is a increase in the uncertainty, that is:

- Before: Y_{t+1}^{hU1} and Y_{t+1}^{lU1}
- After: Y_{t+1}^{hU2} and Y_{t+1}^{lU2}

Since there is an increasing uncertainty, $Y_{t+1}^{hU1} < Y_{t+1}^{hU2}$ and $Y_{t+1}^{lU1} > Y_{t+1}^{lU2}$. Then the corresponding consumption is $C_{t+1}^{hU1} < C_{t+1}^{hU2}$ and $C_{t+1}^{lU1} > C_{t+1}^{lU2}$. That is $C_{t+1}^{hU2} - C_{t+1}^{lU2} > C_{t+1}^{hU1} - C_{t+1}^{lU1}$.

We want to see the change in current savings because of the increasing uncertainty, so we have to fix the expectation of the future consumption at the same level $E(C_{t+1})$. See the graph:



Since $u'''(c) > 0$, when $C_{t+1}^{hU2} - C_{t+1}^{lU2} > C_{t+1}^{hU1} - C_{t+1}^{lU1}$, we have $E^{U2}(u'(C_{t+1})) > E^{U1}(u'(C_{t+1}))$. By Euler equation, $u'(C_t^{U2}) > u'(C_t^{U1})$. Since $u''(c) < 0$, we have $C_t^{U2} < C_t^{U1}$. Given the budget constraint for the first period, $C_t + S_t = Y_t$, we have $CSU2_t > S_t^{U1}$. The increasing uncertainty causes the precautionary savings increasing.

2 Optimal Growth Model

2.1 Connect with Solow model

Consumer side: Euler equation:

$$\frac{u'(C_t)}{\beta u'(C_{t+1})} = 1 + r_t$$

In the log case, $u(C_t) = \ln C_t$, $u'(C_t) = \frac{1}{C_t}$, so we can rewrite the Euler equation as:

$$\frac{C_{t+1}}{C_t} = \beta(1 + r_t)$$

r_t is the interest rate on the savings. Household's savings is right the investment in producer's problem, which is converted into capital for production. So, household's savings interest rate r_t equals the capital return R_{t+1} in the producer's problem. So,

$$\frac{C_{t+1}}{C_t} = \beta(1 + R_{t+1})$$

From the FOC of producer's problem:

$$R_{t+1} = \alpha A_{t+1} K_{t+1}^{\alpha-1}$$

So,

$$\frac{C_{t+1}}{C_t} = \beta(1 + \alpha A_{t+1} K_{t+1}^{\alpha-1})$$

Producer's problem: Recall the Cobb-Douglas production function:

$$Y_t = A K_t^{\alpha} N_t^{1-\alpha}$$

The law of motion:

$$K_{t+1} = (1 - \delta) K_t + I_t$$

Resource constraint:

$$C_t + I_t = Y_t$$

For simplicity, here we assume $\delta = 0$ and $N_t = 1$. Then the production function and the law of motion becomes:

$$Y_t = A_t K_t^{\alpha}$$

$$K_{t+1} = K_t + I_t$$

Combine the resource constraint and production function:

$$I_t = K_{t+1} - K_t$$

$$Y_t - C_t = K_{t+1} - K_t$$

$$A_t K_t^{\alpha} - C_t = K_{t+1} - K_t$$

Combine the producer and consumer's problem

$$\frac{C_{t+1}}{C_t} = \beta\left(1 + \alpha A K_{t+1}^{\alpha-1}\right)$$

$$K_{t+1} - K_t = A_t K_t^{\alpha} - C_t$$

2.2 Solve the steady state

From $\frac{C_{t+1}}{C_t} = \beta\left(1 + \alpha A K_{t+1}^{\alpha-1}\right)$, at steady state, $C_{t+1} = C_t$:

$$1 = \frac{C_{t+1}}{C_t} = \beta\left(1 + \alpha A K^{*\alpha-1}\right)$$

$$\beta\left(1 + \alpha A K^{*\alpha-1}\right) = 1$$

, which gives use the steady state capital level K^*:

$$K^* = \left(\frac{\beta}{1-\beta}\alpha A\right)^{\frac{1}{1-\alpha}}$$

From $K_{t+1} - K_t = A_t K_t^\alpha - C_t$, since at steady state, the capital K^* doesn't change, we have:

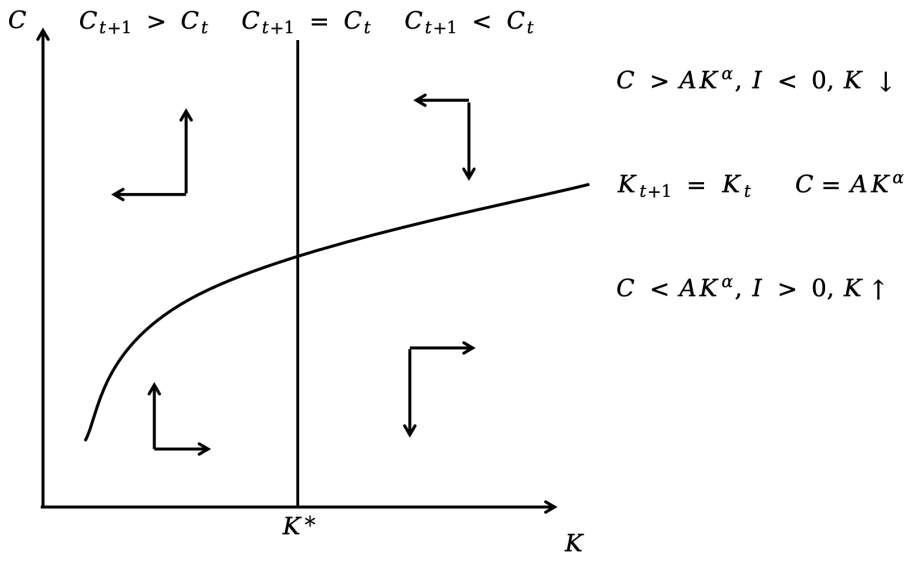
$$AK^{*\alpha} - C^* = 0$$

$$C^* = AK^{*\alpha} = A^{\frac{1}{1-\alpha}}\left(\frac{\beta}{1-\beta}\alpha\right)^{\frac{\alpha}{1-\alpha}}$$

Since at steady state, capital doesn't change, so there is no investment or savings in this economy, $I^* = 0$. By resource contraint, $C_t + I_t = Y_t$:

$$Y^* = C^* = A^{\frac{1}{1-\alpha}}\left(\frac{\beta}{1-\beta}\alpha\right)^{\frac{\alpha}{1-\alpha}}$$

2.3 Dynamic analysis



3 Appendix

3.1 Solve the model with uncertainty

Given $\mathbb{P}(Y_{t+1} = Y_{t+1}^h) = \frac{1}{2}$ and $\mathbb{P}(Y_{t+1} = Y_{t+1}^l) = \frac{1}{2}$, the maximization problem of household is:

$$\max_{C_t, C_{t+1}} u(C_t) + \underbrace{\frac{1}{2}u(C_{t+1}^h) + \frac{1}{2}u(C_{t+1}^l)}_{E_t u(C_{t+1})}$$

s.t.

$$C_t + S_t = Y_t$$
$$C_{t+1}^h = Y_{t+1}^h + S_t$$
$$C_{t+1}^\ell = Y_{t+1}^\ell + S_t$$

Rearrange the constraint, we have two constraints:

$$C_t + C_{t+1}^h = Y_t + Y_{t+1}^h$$

$$C_t + C_{t+1}^l = Y_t + Y_{t+1}^l$$

Then we have the unconstrained maximization problem:

$$\max_{C_t, C_{t+1}} u(C_t) + \frac{1}{2}u(Y_t + Y_{t+1}^h - C_t) + \frac{1}{2}u(Y_t + Y_{t+1}^l - C_t)$$

Take FOC wrt C_t:

$$u'(C_t) = \frac{1}{2}u'(C_{t+1}^h) + \frac{1}{2}u'(C_{t+1}^l)$$

Generally, we can write the Euler equation as:

$$u'(C_t) = E_t(u'(C_{t+1}))$$

3.2 Sum of Geometric Sequence

$g(x) = 1 + x + x^2 + ... + x^T = \frac{1-x^{T+1}}{1-x}$

3.3 Multi-Period Optimal Consumption

Household's maximization problem:

$$\max_{C_t, C_{t+1}, ...} u(C_t) + \beta u(C_{t+1}) + \beta^2 u(C_{t+2}) + \beta^3 u(C_{t+3}) + \cdots + \beta^T u(C_{t+T})$$

$$C_t + \frac{C_{t+1}}{1+r} + \frac{C_{t+2}}{(1+r)^2} + \cdots + \frac{C_{t+T}}{(1+r)^T} = Y_t + \frac{Y_{t+1}}{1+r} + \frac{Y_{t+2}}{(1+r)^2} + \cdots + \frac{Y_{t+T}}{(1+r)^T}$$

The Euler equation is given by:

$$\forall t, u'(C_t) = \beta(1+r)u'(C_{t+1})$$

So, if $\beta(1+r) = 1$, $u'(C_t) = u'(C_{t+1})$, then $C_t = C_{t+1}$.

Marginal propensity to consume: how much increase in income could be converted into consumption.

Examples on slides: solve the utility maximization problem, we are able to find:

$$C_t = \frac{1}{1+\beta+\beta^2+\beta^3+...+\beta^T}\left[Y_t + \frac{Y_{t+1}}{(1+r)} + ... + \frac{Y_{t+T}}{(1+r)^T}\right]$$

Given $\beta(1+r)=1$, thus $\beta = \frac{1}{1+r}$:

$$C_t = \frac{1}{1+\beta+\beta^2+\beta^3+...+\beta^T}\left[Y_t + \beta Y_{t+1} + ... + \beta^T Y_{t+T}\right]$$

Apply the math rule:

$$C_t = \frac{1-\beta}{1-\beta^{T+1}}\left[Y_t + \beta Y_{t+1} + ... + \beta^T Y_{t+T}\right]$$

Since $\beta(1+r)=1$, then $C_t = C_{t+1}, \forall t$:

$$\bar{C} = \frac{1-\beta}{1-\beta^{T+1}}\left[Y_t + \beta Y_{t+1} + ... + \beta^T Y_{t+T}\right]$$

Then we can calculate the MPC by taking the first order derivative of $\bar{C}$ wrt Y_t.