1 Understand Saddle Path

1.1 Definition of saddle path

We want to know, given a particular level of capital K_t, what level of consumption C_t should the household choose?How will the economy gradually converge to the steady state.
First, we should understand what is the saddle path.
The saddle path describes how the economy gradually converges to the steady state when it is not already there. It is a concept related to **time**. So, we need three equations to determine the saddle path: **Euler equation, law of motion, and lifetime budget constraint**.
Suppose the utility function takes the log form: $U(c) = \ln C$. Let's break the model step by step:
Euler equation:

$$\frac{C_{t+1}}{C_t} = \beta(1 + \alpha A K_{t+1}^{\alpha-1})$$

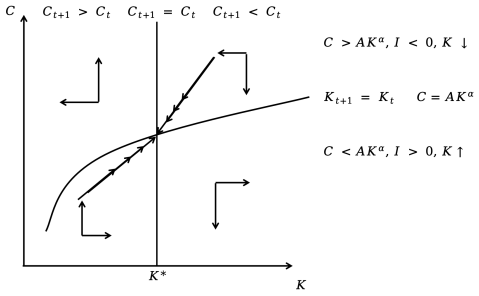
Euler equation tells us: given today's consumption C_t and the interest rate $\alpha A K_{t+1}^{\alpha-1}$, what is the level of consumption tomorrow C_{t+1}.
Law of motion:

$$K_{t+1} - K_t = AK_t^{\alpha} - C_t$$

Law of motion tells us: given today's capital K_t and consumption C_t, what is the level of capital tomorrow K_{t+1}. Law of motion always holds, no matter whether the consumption and savings decision is optimal.
Question: For a given capital level K_t, we know the level of capital tomorrow K_{t+1} by law of motion. We also know how much to consume tomorrow C_{t+1}. Then how much should the household consume today? What is the level of C_t?
Solution: We need the **lifetime budget constraint**.

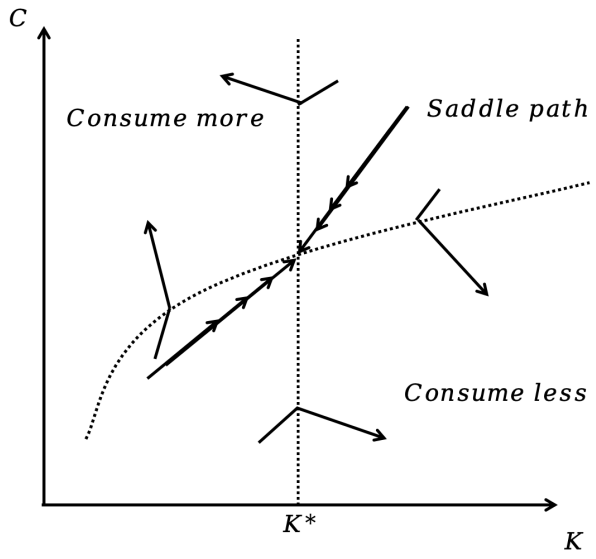
$$C_t + \frac{1}{1+R_{t+1}} C_{t+1} + \frac{1}{(1+R_{t+1})(1+R_{t+2})} C_{t+2} + \cdots = Y_t + \frac{1}{1+R_{t+1}} Y_{t+1} + \frac{1}{(1+R_{t+1})(1+R_{t+2})} Y_{t+2} + \ldots$$

The graph below is the saddle path:



1.2 Intuition of saddle path

Given a particular level of capital K , only the consumption level chosen by the household **on the saddle path** allows the economy to converge to the steady state.



- Points above the saddle path: The present discounted value of consumption exceeds that of income.

  $$C_t+\frac{1}{1+R_{t+1}}C_{t+1}+\frac{1}{(1+R_{t+1})(1+R_{t+2})}C_{t+2}+\cdots > Y_t+\frac{1}{1+R_{t+1}}Y_{t+1}+\frac{1}{(1+R_{t+1})(1+R_{t+2})}Y_{t+2}+\ldots$$

  The economy will eventually move in the upper-left direction.

- Points below the saddle path. The present discounted value of consumption is below that of income.

  $$C_t+\frac{1}{1+R_{t+1}}C_{t+1}+\frac{1}{(1+R_{t+1})(1+R_{t+2})}C_{t+2}+\cdots < Y_t+\frac{1}{1+R_{t+1}}Y_{t+1}+\frac{1}{(1+R_{t+1})(1+R_{t+2})}Y_{t+2}+\ldots$$

  The economy will eventually move in the lower-right direction.

2 Change in the parameters

Steps to solve these questions:

- Step 1: Solve the optimal growth model, especially the steady state level of capital
- Step 2: See if there is a change in the steady state level of capital K^*
- Step 3: See if there is a change in the relationship of consumption and capital at the steady state $C = AK^\alpha$
- Step 4: Plot the new saddle path. Check if the change of consumption as soon as the change occurs.

An increase in A Follow the steps to plot the new graph:

- Step 1: Solve the optimal growth model:
  The Euler equation is given by:

  $$\frac{C_{t+1}}{C_t} = \beta(1+\alpha AK_{t+1}^{\alpha-1})$$

At the steady state, $C_{t+1} = C_t$, so:

$$1 = \frac{C_{t+1}}{C_t} = \beta(1 + \alpha A K^{*\alpha-1})$$

The capital at steady state is given by:

$$K^* = \left(\frac{\beta}{1-\beta}\alpha A\right)^{\frac{1}{1-\alpha}}$$

The law of motion is given by:

$$K_{t+1} - K_t = AK_{t+1}^{\alpha} - C_t$$

At the steady state, $K_{t+1} = K_t$, so:

$$C = AK^{\alpha}$$

- Step 2: See if there is a change in the steady state level of capital K^*

$$K^* = \left(\frac{\beta}{1-\beta}\alpha A\right)^{\frac{1}{1-\alpha}}$$

K^* is increasing in A. So the vertical line in the graph will shift to the right.

- Step 3: See if there is a change in the relationship of consumption and capital at the steady state $C = AK^{\alpha}$:
When A increases, consumption at all capital level will increase. So the whole curve shifts up.

- Step 4: Plot the new saddle path. Check if the change of consumption as soon as the change occurs:
Change in consumption as soon as a sudden increase in A occurs is ambiguous. Recall the Solow model:

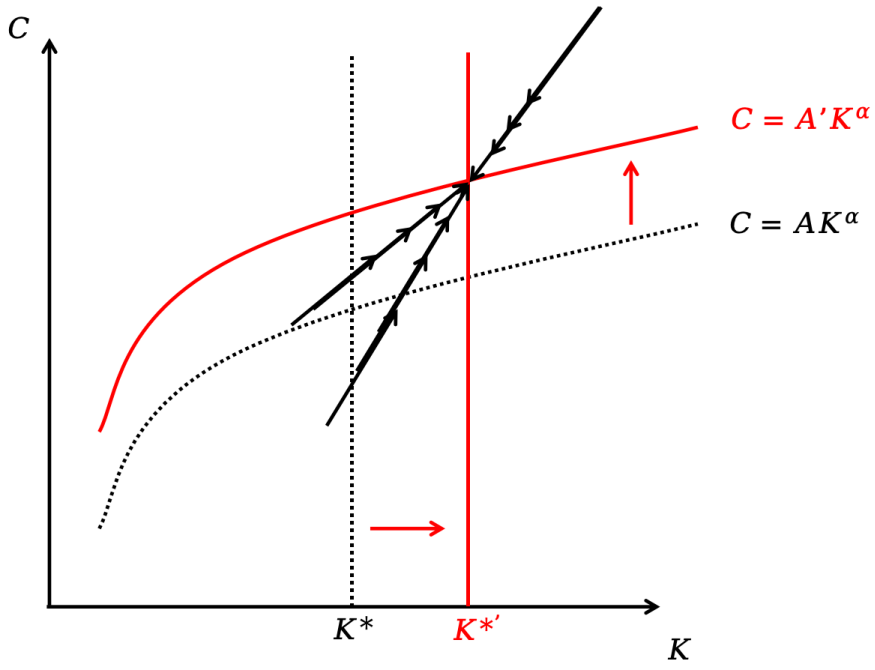
$$C_t = (1-s)AY_t$$

(Note that the savings rate is not a constant now! It changes!)

  - The productivity goes up, the economy is able to produce more efficiently. This will cause the consumption C_t to go up.
  - A higher A requires a higher capital K^* at the new steady state. To raise the capital stock, households have to save more to accumulate the capital stock. This will cause the consumption C_t to go down.

  Whether the consumption C_t will go up or go down depends on whether the households want to reach the new steady state quickly.

  - (Large β, households are patient with consumption)If the households want to reach the new steady state quickly, they have to accumulate capital stock quickly. They will raise the savings much, which will cause the consumption C_t to go down at first.
  - (Small β, household are impatient with consumption)Households won't raise savings much. The consumption C_t will increase since the productivity goes up.



3 Neoclassical Model

The classical dichotomy assumption allows us to analyze the real and nominal variables separately. We will focus on the real variables today(goods market equilibrium).

Real block

$$
\begin{aligned}
C_t &= C^d(Y_t - G_t, Y_{t+1} - G_{t+1}, r_t) \\
N_t &= N^s(w_t, \theta_t) \\
N_t &= N^d(w_t, A_t, K_t) \\
I_t &= I^d(r_t, A_{t+1}, K_t) \\
Y_t &= A_t F(K_t, N_t) \\
Y_t &= C_t + I_t + G_t
\end{aligned}
$$

Nominal block

$$
\begin{aligned}
M_t &= P_t M^d(r_t + \pi^e_{t+1}, Y_t) \\
r_t &= i_t - \pi^e_{t+1}
\end{aligned}
$$

3.1 Assumptions

- Classical dichotomy: This allows us to analyze the real variables and nominal variables separately.

- MPC is lower than 1:

$$MPC = \frac{\partial C^d}{\partial Y_t} < 1$$

  So, the slope of Y^d is lower than 1

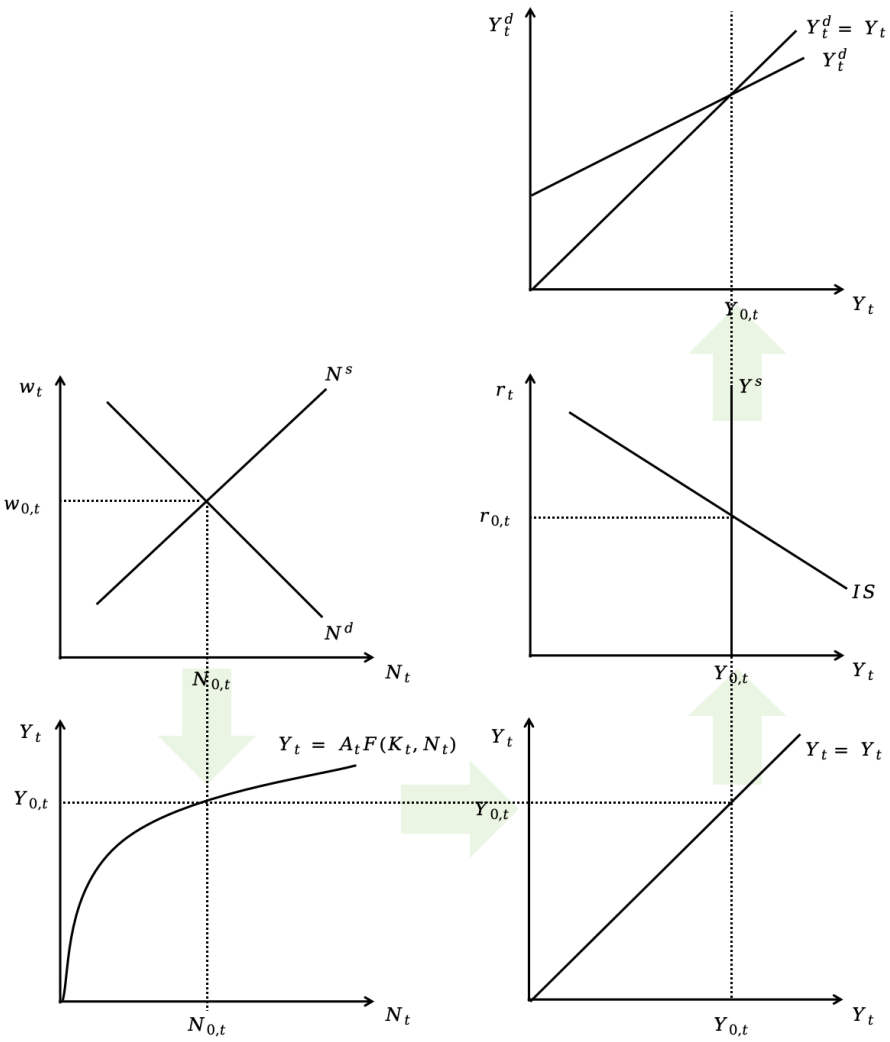
- Substitution effect dominates income effect:
  - $C^d(Y_t - G_t, Y_{t+1} - G_{t+1}, r_t)$, consumption C^d decreases when real interest rater_t goes up.(Recall the interest rate represents the relative price of C_t relative to C_{t+1})
  - $N^s(w_t, \theta_t)$, labor supply N^s increases when real wage w_t decreases. Since the real wage is the opportunity cost for not working.

3.2 Properties

- When real interest rate goes up, the investment will go down. Because the interest rate is the "cost" of investment. So, the IS curve is downward sloping.

3.3 Steps to plot a series of graphs

- Plot the labor market(labor against wage) supply and demand curve, the intersection determines the equilibrium labor N
- Plot the output against labor curve. Given the equilibrium labor N, this curve represents the output supply Y^s.
- Use the 45-degree line to map Y^s to the IS curve graph
- Plot the IS curve(interest rate r_t against output Y_t). Given $Y_t = Y_s$, the intersection of the IS curve and the $Y_t = Y_s$ line determines the equilibrium interest rate r.
- The condition $Y_t = Y_s$ helps us identify the output demand curve Y^d



4 Shocks in the neoclassical model

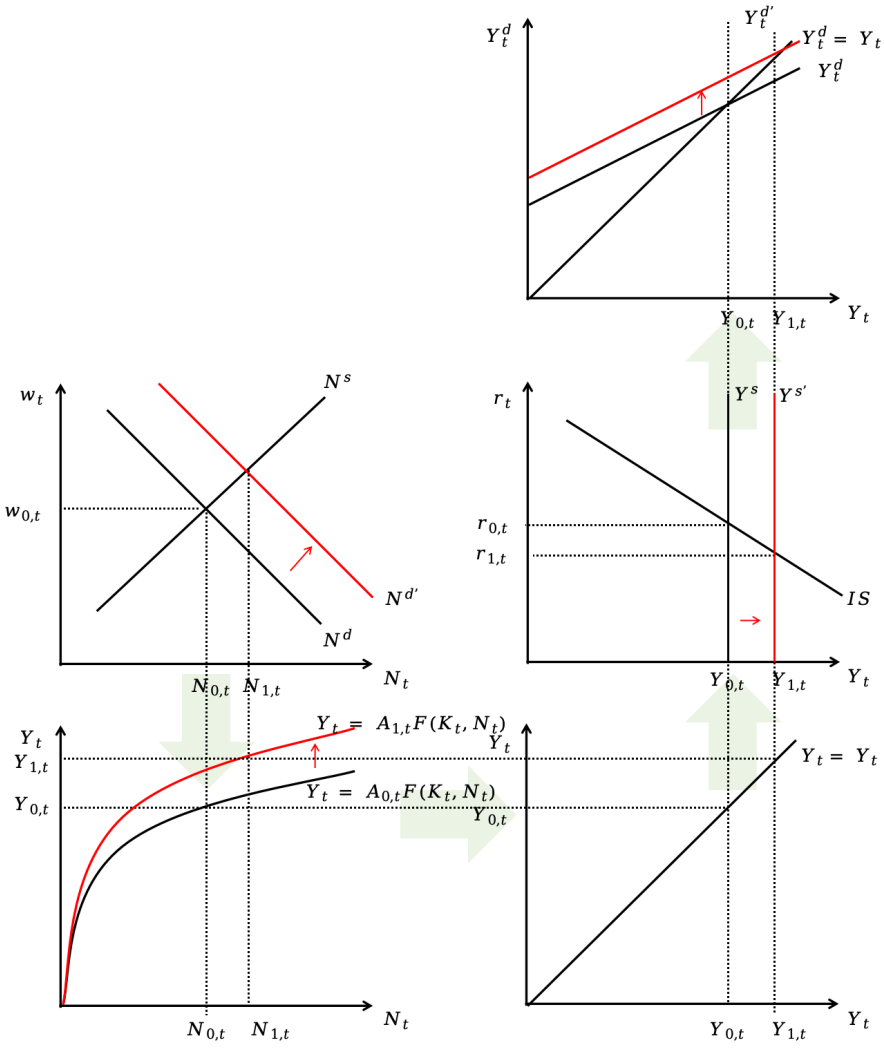
Sudden increase in A: $A_1 > A_0$

4.1 Steps

- Step 1: write down all the equations of neoclassical model:

$$
\begin{aligned}
C_t &= C^d(Y_t - G_t, Y_{t+1} - G_{t+1}, r_t) \\
N_t &= N^s(w_t, \theta_t) \\
N_t &= N^d(w_t, A_t, K_t) \\
I_t &= I^d(r_t, A_{t+1}, K_t) \\
Y_t &= A_t F(K_t, N_t) \\
Y_t &= C_t + I_t + G_t
\end{aligned}
$$

- Step 2: Check whether the change will shift the curves in the following order:
  - Check the labor market:
    - Labor supply $N^s(w_t, \theta_t)$: no change
    - Labor demand $N^d(w_t, A_t, K_t)$: shifts to the right. Intuition: producers are more productive, so they need more workers.
  - Check the production function: $Y_t = A_t F(K_t, N_t)$. It shifts upward when the TFP A increases
  - Check whether the IS curve shifts: it won't shift when the current productivity A_t increases.
- Step 3: Check whether these curve shifts will cause a change in the output supply Y^s. If so, the graph in the right column will change accordingly.



5 Summary of exogenous shocks on endogenous variables [1]

Variable	$\uparrow A_t$	$\uparrow A_{t+1}$	$\uparrow G_t$	$\uparrow M_t$	$\uparrow \pi^e_{t+1}$
Y_t	+	0	0	0	0
C_t	+	?	-	0	0
I_t	+	?	-	0	0
N_t	+	0	0	0	0
w_t	+	0	0	0	0
r_t	-	+	+	0	0
i_t	-	+	+	0	+
P_t	-	+	+	+	+

Table 1: Impact of Exogenous Shocks on Economic Variables

[1] Table by Andy Lehrer