# 1 Overview

- The review sheet provides an outline of the models we have covered so far.
- Note that this is **not comprehensive**.
- Enjoy!

# 2 Preparation

## 2.1 Homogeneous of degree $\theta$

A function $F(K, N)$ is **homogeneous of degree** $\theta$ if for all $\lambda > 0$:

$$F(\lambda K, \lambda N) = \lambda^{\theta} F(K, N)$$

.

### 2.1.1 Review checklist

Prove the Cobb-Douglas production function is homogeneous of degree 1. (Handout 2)

## 2.2 Euler theorem

$$\rho F(K_t, N_t) = F_K(K_t, N_t)K_t + F_N(K_t, N_t)N_t$$

### 2.2.1 Review checklist

Assume $\rho = 1$(homogeneous of degree 1), prove the economic profit is zero in the firm's problem. (Handout 2)

## 2.3 Important properties of production function

Important properties:

- $F_K > 0$, $F_N > 0$
- $F_{KK} < 0$, $F_{NN} < 0$
- $F_{KN} > 0$
- $F(K, 0) = 0$, $F(0, N) = 0$

### 2.3.1 Review checklist

Interpret the intuition of these important properties.

## 2.4 Important properties of production function in terms of capital per worker

- $k_{t+1}$ is monotonically increasing with $k_t$: $\frac{dk_{t+1}}{dk_t} = sAf'(k_t) + (1 - \delta) > 0$.
- $f(k)$ is monotonically increasing and is concave: $f' > 0$, $f'' < 0$
- Inada conditions: $\lim_{k_t \to 0} f'(k_t) = \infty$ and $\lim_{k_t \to \infty} f'(k_t) = 0$

#### 2.4.1 Review checklist

How will these help to plot the law of motion?

### 2.5 Key properties of utility function

Household's utility function:

$$U = u(C_t) + \beta u(C_{t+1})$$

- $\beta < 1$
- $u'(c) > 0$
- $u''(c) < 0$

### 2.6 Sum of Geometric Sequence

$g(x) = 1 + x + x^2 + ... + x^T = \frac{1-x^{T+1}}{1-x}$

## 3 Solow Model

### 3.1 Outline

Key Equations(aggregate) of Solow Model:

- Production function: $Y_t = A_t F(K_t, N_t)$
- Resource constraint: $Y_t = C_t + I_t$
- Law of motion: $K_{t+1} = I_t + (1-\delta)K_t$
- Savings: $I_t = sY_t$
- FOCs: $w_t = AF_N(K_t, N_t)$, $r_t = AF_K(K_t, N_t)$

Key variables in terms of "per worker":

- $k_{t+1} = sAf(k_t) + (1-\delta)k_t$
- $y_t = Af(k_t)$
- $c_t = (1-s)Af(k_t)$
- $i_t = sAf(k_t)$
- $r_t = Af'(k_t)$
- $w_t = Af(k_t) - Af'(k_t)k_t$

Steady state(no growth):

- Steady state level of capital per worker is given by: $k^* = sAf(k^*) + (1-\delta)k^*$. Given the steady state level of capital, other important variables are given by:
  - $y^* = Af(k^*)$

- $c^* = (1-s)Af(k^*)$
- $i^* = sAf(k^*)$
- $r^* = Af'(k^*)$
- $w^* = Af(k^*) - Af'(k^*)k^*$

Solve the graph questions(at first the economy is at steady state, then there is a sudden change(sudden increase or decrease in $s, A, K, N$), solve the new steady state then plot how will the economy converge to the new steady state)

- Step 1: Solve for the old and new steady states. Write the equations for key variables explicitly.
- Step 2: Determine capital per worker at both steady states. Check if the law of motion curve shifts.
- Step 3: Plot key variables at both steady states.
- Step 4: Use the equations to check for jumps or drops in key variables. Plot their transition from the old to the new steady state, keeping in mind that the marginal output of capital diminishes.

### 3.2 Review checklist

- Solve the first order condition of firm's problem. (Handout 2)
- Practice writing the key variables in terms of capital per worker. (Handout 3)
- Solve the steady state. (Handout 3)
- Practice: If there is a change in $s$, $A$, $K$, or $N$, plot the old and new steady states along with the transition path.

## 4 Solow Growth Model

### 4.1 Outline

Key variables in terms of "per effective worker":

- $\hat{k}_{t+1} = \frac{1}{(1+z)(1+n)}\left[sAf(\hat{k}_t) + (1-\delta)\hat{k}_t\right]$
- $\hat{y}_t = Af(\hat{k}_t)$
- $\hat{i}_t = sAf(\hat{k}_t)$
- $\hat{c}_t = (1-s)Af(\hat{k}_t)$
- Special cases: no need to write in terms of "per effective worker":
  - $R_t = Af'(\hat{k}_t)$
  - $w_t = AZ_t(f(\hat{k}_t) - f'(\hat{k}_t)\hat{k}_t)$

Steady state:

- Steady state level of capital per effective worker is given by: $\hat{k}^* = \frac{1}{(1+z)(1+n)}\left[sAf(\hat{k}^*) + (1-\delta)\hat{k}^*\right]$. Given $\hat{k}^*$, other important variables are given by:
  - $\hat{y}^* = Af(\hat{k}^*)$
  - $\hat{i}^* = sAf(\hat{k}^*)$
  - $\hat{c}^* = (1-s)Af(\hat{k}^*)$
  - $R^* = Af'(\hat{k}^*)$
  - $w^* = AZ^*(f(\hat{k}^*) - f'(\hat{k}^*)\hat{k}^*)$

There are two types of graphs:

- Plot the key variables in terms of "per effective worker." This is mostly similar to the standard Solow model (no growth).
- Plot the key variables in terms of "per worker." Note that the slopes are now nonzero since there is growth at the steady state. We plot the **log form** of key variables to illustrate the constant growth rate more clearly.

### 4.2 Review checklist

- Practice writing the key variables in terms of capital per effective worker. (Handout 4)
- Solve the steady state. (Handout 4)
- What is the growth rate of the key variables at the steady state? What is the difference between the growth rate in "per worker" terms and aggregate terms? (Handout 4)
- Practice: If there is a change in $s$, $A$, $Z$,$K$, or $N$, plot the old and new steady states along with the transition path. Plot the change in the growth rate of the key variables.

## 5 Optimal consumption

### 5.1 Outline

$$\max_{C_t, C_{t+1}} u(C_t) + \beta u(C_{t+1})$$

s.t.

$$C_t + S_t = Y_t$$
$$C_{t+1} = Y_{t+1} + (1+r_t)S_t$$

The Euler equation:

$$u'(C_t) = \beta(1+r_t)u'(C_{t+1})$$

Graph questions:

- The single budget constraint represents the budget line.
- The utility function $U = u(C_t) + \beta u(C_{t+1})$ represents the indifference curve.
- The tangency point between the budget line and an indifference curve represents the optimal choice.

Model with uncertainty; The maximization problem of household is:

$$\max_{C_t, C_{t+1}} u(C_t) + \underbrace{\frac{1}{2}u(C_{t+1}^h) + \frac{1}{2}u(C_{t+1}^l)}_{E_t u(C_{t+1})}$$

s.t.

$$C_t + S_t = Y_t$$
$$C_{t+1}^h = Y_{t+1}^h + S_t$$
$$C_{t+1}^\ell = Y_{t+1}^\ell + S_t$$

Precautionary savings:

$$\text{Positive precautionary savings} \Leftrightarrow u'''(.) > 0$$

### 5.2 Review Checklist

- Solve for the single constraint (Handout 5).
- Interpret the intuition behind the Euler equation (Handout 5).
- Plot the budget line and the indifference curve. What does it mean when the budget line is tangent to a certain indifference curve? (Handout 5)
- What is the definition of marginal propensity of consumption(MPC)?(Lecture 07 slides)
- Practice: solve the multi-period version of optimal consumption model. (Lecture 07 slides)
- Practice: Graphically analyze the effects of changes in $Y_t$, $Y_{t+1}$, and $r$. Whether it shift or rotate the budget line?
- What are the income effect and substitution effect? Which one dominates the other? (Handout 5)
- Practice: Graphically prove that when $u'''(.) > 0$, there will be precautionary savings as uncertainty increases. Hint: Use the Euler equation. (Handout 5)

## 6 Optimal Growth Model

### 6.1 Outline

Three important equations:

- Euler equation(from consumer's problem): $\frac{C_{t+1}}{C_t} = \beta\left(1 + \alpha A K_{t+1}^{\alpha-1}\right)$
- Law of motion(from producer's problem): $K_{t+1} - K_t = A_t K_t^\alpha - C_t$
- Life time budget constraint:

$$C_t + \frac{1}{1+R_{t+1}}C_{t+1} + \frac{1}{(1+R_{t+1})(1+R_{t+2})}C_{t+2} + \cdots = Y_t + \frac{1}{1+R_{t+1}}Y_{t+1} + \frac{1}{(1+R_{t+1})(1+R_{t+2})}Y_{t+2} + \ldots$$

Steady state:

- Euler equation: $C_{t+1} = C_t$ gives $K^*$
- Law of motion: $K_{t+1} = K_t$ gives the curve $C^* = AK^{*\alpha}$(when the production function is Cobb-Douglas form)
- Life time budget constraint gives the saddle path

Analyze the new steady state and the saddle path when there is a change in the parameter:

- Step 1: Solve the optimal growth model, especially the steady state level of capital
- Step 2: See if there is a change in the steady state level of capital $K^*$
- Step 3: See if there is a change in the relationship of consumption and capital at the steady state $C = AK^{\alpha}$
- Step 4: Plot the new saddle path. Check if the change of consumption as soon as the change occurs(jump or drop).

### 6.2 Review checklist

- How to solve the optimal growth model?(Handout 6)
- Use the graph to interpret the features of the four quadrants formed by the curves $K = K^*$ and $C^* = AK^{*\alpha}$. (Handout 6)
- What is the intuition behind the saddle path? In which direction will the economy converge if the point is above or below the saddle path? (Handout 7)
- What if there is a change in $A$, $\beta$? Use the graph to analyze. (Handout 7 & Lecture 08 slides)

## 7 Neoclassic Model

### 7.1 Outline

Key equations:

$$\begin{aligned} C_t &= C^d(Y_t - G_t, Y_{t+1} - G_{t+1}, r_t) \\ N_t &= N^s(w_t, \theta_t) \\ N_t &= N^d(w_t, A_t, K_t) \\ I_t &= I^d(r_t, A_{t+1}, K_t) \\ Y_t &= A_t F(K_t, N_t) \\ M_t &= P_t M^d(r_t + \pi^e_{t+1}, Y_t) \\ r_t &= i_t - \pi^e_{t+1} \end{aligned}$$

Key assumptions:

- Classical dichotomy
- MPC is lower than 1
- Substitution effect dominates income effect

Steps to plot the 5 graphs:

- Plot the labor market(labor against wage) supply and demand curve, the intersection determines the equilibrium labor $N$
- Plot the output against labor curve. Given the equilibrium labor $N$, this curve represents the output supply $Y^s$.
- Use the 45-degree line to map $Y^s$ to the IS curve graph
- Plot the IS curve(interest rate $r_t$ against output $Y_t$). Given $Y_t = Y_s$, the intersection of the IS curve and the $Y_t = Y_s$ line determines the equilibrium interest rate $r$.
- The condition $Y_t = Y_s$ helps us identify the output demand curve $Y^d$

Analyze the shocks in neoclassical model

- Step 1: Check if the shock changes equilibrium labor (determine whether the labor market or labor demand curve shifts).
- Step 2: Check if the shock affects the production function.
- Step 3: Check if the shock shifts the IS curve.
- Step 4: Changes in output (from the production function graph) and the equilibrium interest rate (from the IS curve graph) indicate whether the money demand curve shifts.

### 7.2 Review checklist

- What do the key assumptions tell us? (Handout 7)
- Why the IS curve is downward sloping? (Lecture 09 slides)
- Practice: Plot 5 graphs by yourself
- Why the money supply curve is vertical? Why the money demand curve is upward sloping? (Lecture 09 slides)
- Practice the cases on Lecture 09 slides.